CCSU Regional Math Competition, 2016

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

Problem 1. For each real number $t \in [-1, 1]$ let P_t be the parabola in the xy-plane that has axis parallel to the y-axis, passes through the points $(0, 0)$ and $(4, t)$, and has a tangent line with a slope $t - 1$ at the point $(4, t)$. Find the smallest possible y-coordinate for the vertex of P_t.

Problem 2. Inside a square of side 2 there are 7 polygons each of area 1. Show that there are 2 polygons that overlap over a region of area at least $\frac{1}{7}$.

Problem 3. Consider two matrices A $(m \times n)$ and B $(n \times m)$ with real entries, such that $m \geq n \geq 2$. Assume there exist an integer $k \geq 1$ and real numbers $a_0, a_1, ..., a_k$ such that

$$a_k(AB)^k + a_{k-1}(AB)^{k-1} + \cdots + a_2(AB)^2 + a_1(AB) + a_0 I_m = O_m,$$

$$a_k(BA)^k + a_{k-1}(BA)^{k-1} + \cdots + a_2(BA)^2 + a_1(BA) + a_0 I_n \neq O_n,$$

where I_m, I_n are the identity matrices and O_m, O_n are the zero matrices of the corresponding sizes. Prove that $a_0 = 0$.

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Part II

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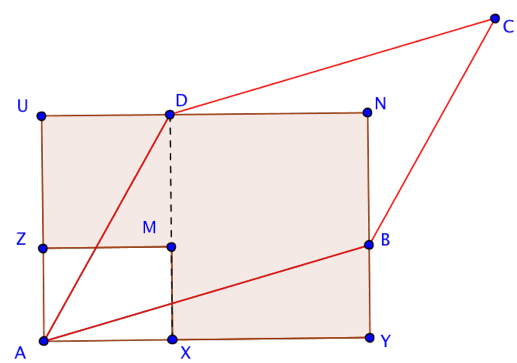
Problem 4. Show that the area of the region $MXYNUZ$ equals the area of the parallelogram $ABCD$ where the lines $\overleftrightarrow{AY}$, $\overleftrightarrow{BZ}$, $\overleftrightarrow{NU}$ are perpendicular to the line $\overleftrightarrow{AU}$ and the lines $\overleftrightarrow{DX}$, $\overleftrightarrow{NY}$ are perpendicular to the line $\overleftrightarrow{AY}$. The segments $\overline{DX}$, $\overline{BZ}$ meet at the point M and their endpoints are on the sides of the polygon $AYNU$ as in the figure.



Problem 5. Compute the integral

$$\int_0^{\pi/4} \ln(1 + \tan x)dx.$$

Problem 6. Let f be the function defined recursively by $f(0) = 1$ and $f(n) = 1 + nf(n - 1)$ for each positive integer n. Find the smallest prime divisor of $f(4 \times 30 + 2016)$.