CCSU Regional Math Competition, 2015

SOLUTIONS I

Problem 1. Suppose the vertices of a hexagon are labeled by the integers 1 through 6, each used just once. For each edge, the absolute difference of the labels at the endpoints is an element of $\{2, 3, 4\}$. Furthermore, the sum of labels at diametrically opposite vertices is never 7. If one edge is chosen at random, what is the probability that the absolute difference of its endpoint labels is 2?

Solution. We can systematically work out all the possible labelings. We start at the vertex labeled 1, and work sequentially around the perimeter of the hexagon, always respecting the 'absolute difference' constraint. (It makes no difference if we proceed clockwise or counter-clockwise.) If we view the sequence 1,2,3,4,5,6 cyclically, then the constraint simply says that adjacent vertices cannot be labeled by consecutive integers.

The initial label 1 can be followed by either 3, 4, or 5; we track each of these cases separately. 13 can be followed by 5 or 6, yielding two sub-cases. 135 can be followed only by 2, and then 1352 can be extended to just one full labeling, 135264. 136 can be followed by 2 or 4. 1362 leads nowhere, because the two remaining integers are consecutive; 1364 extends to just one full labeling, 136425. This completes the analysis of case 13. Cases 14 and 15 can be followed through in exactly the same fashion. From 14 we find 142635, which violates the 'opposite vertices' constraint, and 146253, which is just the mirror image of the labeling 135264 found earlier. From 15 we again find two new labelings, but both are mirror images of labelings already found.

Thus there are just two fundamentally different valid labelings: 135264 and 136425. Each has the same collection of absolute differences, namely 2,2,2,3,3,4. Hence, a randomly chosen edge will yield a 2 with probability $\frac{1}{2}$.

Problem 2. Let R be the operation on 2-by-2 matrices that 'rotates' the entries 90° clockwise. That is, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define $A^R = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$. Find all matrices having real or complex entries and satisfying $A^2 = A^R$.

Solution I. First, we observe that $\det A = ad - bc$, while $\det A^R = bc - ad = -\det A$, so from the fact that $\det(A^2) = (\det A)^2$ and the equation, we obtain the identity

$$\det A \cdot (\det A + 1) = 0.$$

Hence the determinant has to either be 0 or -1, therefore $ad = bc$ or $ad = bc - 1$. We will use these later.

Next, we observe that $A^2 = A^R$ produces the following system of equations:

$$\begin{cases} a^2 + bc = c & (1) \\ ab + bd = a & (2) \\ ac + dc = d & (3) \\ bc + d^2 = b & (4) \end{cases}$$

Subtract (4) from (1) and (3) from (2):

$$a^2 - d^2 = c - b \Rightarrow (a-d)(a+d) = c - b$$
$$ab + bd - ac - cd = a - d \Rightarrow (b-c)(a+d) = a - d$$

Replacing $(a - d)$ from the second one and plugging it into the first one we get:

$$(b-c)[(a+d)^2 + 1] = 0 \qquad (I)$$

Now, we add (2) and (3) and get $(a+d)(b+c) = a + d$, so:

$$(a+d)(b+c-1) = 0 \qquad (II)$$

Therefore we have 4 cases: $b = c$, $a + d = \pm i$, $a = -d$ and $b + c = 1$.

Case I: $b = c$. Then it follows that $a^2 = d^2$, so $a = d$ or $a = -d$. If $a = d$, then $2ab = a$, $2ac = a$ and $a^2 + b^2 = b$. In this case either $a = 0$ or $b = c = \frac{1}{2}$.

When $a = 0$, then $d = 0$, so $b^2 = b$, hence $b = 0$ or $b = 1$. We obtain two matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When $b = c = \frac{1}{2}$, then $a^2 = \frac{1}{4}$, so $a = \pm\frac{1}{2} = d$. This gives two more matrices:

$$A_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A_4 = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

If $a = -d$, then from (2) it follows that $a = 0$, so $d = 0$ and in that case $b^2 = b$, leading to the same matrices as above, A_1 and A_2.

Case II: $a = -d$. From (2) and (3) it follows that $a = d = 0$, so $bc = c = b$, so again $b^2 = b$, giving the already obtained matrices A_1 and A_2.

Case III: $b + c = 1$. By adding (1) and (4), we obtain $a^2 + 2bc + d^2 = 1$, which is the same as $a^2 + 2ad + d^2 + 2bc - 2ad = 1$. If $\det A = -1$, then we deduce that $(a + d)^2 = -1$, which gives the case $a + d = \pm i$ (treated separately). If, on the other hand, $\det A = 0$, then $(a+d)^2 = 1$, so $a+d = \pm 1$. The case $a + d = 1$ gives $b = a$ and $c = d$ (from (2) and (3)), and it follows from (1) and (4) that $b^2 + bc = c$ and $bc + c^2 = b$ and subtracting these gives $(b - c)(b + c + 1) = 0$, so $b = c$ or $b + c = -1$. The second case is impossible, since $b + c = 1$, so we get that $b = c = \frac{1}{2}$ and thus the matrix A_3 is obtained. Similarly, when $a + d = -1$, the matrix A_4 is obtained.

Case IV: $a + d = \pm i$. We treat the case $a + d = i$ first. From (2) and (3), we get that $ib = a$ and $ic = d$, so $b = -ia$ and $c = -id$. Hence $b + c = 1$. From case III, we have that $\det A = -1$ (the other case gave the known matrices). Therefore we have the system

$$\begin{cases} b = -ia \\ c = -id \\ a + d = i \\ bc - ad = 1 \end{cases}$$

It's easy to see that this lead to the equation $2a^2 - 2ia - 1 = 0$, with solution $a = \frac{1}{2}(i + 1)$ or $a = \frac{1}{2}(i - 1)$. So we obtain two matrices:

$$A_5 = \frac{1}{2}\begin{bmatrix} i+1 & 1-i \\ 1+i & i-1 \end{bmatrix} \qquad A_6 = \frac{1}{2}\begin{bmatrix} i-1 & 1+i \\ 1-i & i+1 \end{bmatrix}.$$

The case $a + d = -i$ leads to the fact that $-ib = a$, $-ic = d$, $b = ia$, $c = id$. Moreover, this gives the equation $2a^2 + 2ia - 1 = 0$, so $a = \frac{1}{2}(-i + 1)$ or $a = \frac{1}{2}(-i - 1)$, which creates two more matrices:

$$A_7 = \frac{1}{2}\begin{bmatrix} -i+1 & 1+i \\ 1-i & -i-1 \end{bmatrix} \qquad A_8 = \frac{1}{2}\begin{bmatrix} -i-1 & 1-i \\ 1+i & -i+1 \end{bmatrix}.$$

Solution II. The problem is reduced to solving the system

$$a^2 + bc = c, \qquad ab + bd = a, \qquad ac + dc = d, \qquad bc + d^2 = b.$$

Subtracting the last from the first and the third from the second, we get

$$(a-d)(a+d) = c-b, \qquad (a+d)(b-c) = a-d.$$

This suggests the change of variables

$$a-d = x, \qquad b-c = y, \qquad a+d = u, \qquad b+c = t.$$

In terms of the new variables, the system can be written as

$$xu = -y, \quad yu = x, \quad (t+y)u = x+u, \quad t^2 - y^2 + (x-u)^2 = 2y + 2t.$$

Equivalently,

$$xu = -y, \qquad yu = x, \qquad tu = u, \qquad (t-1)^2 + x^2 + u^2 = y^2 + 1.$$

Starting with the third equation, we have

Case 1. If $u = 0$, we get $x = y = 0$ and $t = 2$ or $t = 0$.
Case 2. If $t = 1$, we get $xu^2 = -x$ and $x^2 + u^2 = x^2u^2 + 1$.
Subcase 2a. If $x = 0$ we get $u = 1$ or $u = -1$ and $y = 0$.
Subcase 2b. If $u^2 = -1$ we get $x^2 = 1$ and $y = -ux$.

Case 1 and Subcase 2a give four real-entry matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}, \qquad \begin{bmatrix} -.5 & .5 \\ .5 & -.5 \end{bmatrix},$$

and Subcase 2b gives four complex-entry matrices:

$$\begin{bmatrix} .5+.5i & .5-.5i \\ .5+.5i & -.5+.5i \end{bmatrix}, \qquad \begin{bmatrix} -.5+.5i & .5+.5i \\ .5-.5i & .5+.5i \end{bmatrix},$$
$$\begin{bmatrix} .5-.5i & .5+.5i \\ .5-.5i & -.5-.5i \end{bmatrix}, \qquad \begin{bmatrix} -.5-.5i & .5-.5i \\ .5+.5i & .5-.5i \end{bmatrix}.$$

Problem 3. Let $y = f(x)$ be a function defined on $[0,1]$. In each of the following cases, find the largest real number B such that the statement

$$\int_0^1 (y'^2 + y)dx \geq B$$

is true for all functions of the type specified:
(a) f is linear, with $f(0) = 0$.
(b) f is continuously differentiable, with $f(0) = 0$.

Solution I. (a) For linear functions $f(x) = mx$, we minimize

$$\int_0^1 (y'^2 + y)dx = \int_0^1 (m^2 + mx)dx = m^2 + m/2$$

where m is the slope of the line. The minimum is achieved when $m = -\frac{1}{4}$, which gives $B = -1/16$.

(b) For continuosuly differentiable functions we integrate by parts

$$\begin{aligned}\int_0^1 ydx &= xy|_{x=0}^{x=1} - \int_0^1 y'xdx = y(1) - \int_0^1 y'xdx \\ &= \int_0^1 y'dx - \int_0^1 y'xdx = \int_0^1 y'(1-x)dx.\end{aligned}$$

Hence, we minimize the integral

$$\int_0^1 (y'^2 + y)dx = \int_0^1 [y'^2 + y'(1-x)]dx.$$

At each x the integrand $y'^2 + (1-x)y'$ is a quadratic function in y' whose minimum is attained (by the standard formula for the vertex of a parabola) at $y' = \frac{1}{2}(x-1)$ and thus,

$$y'^2 + (1-x)y' \geq -\tfrac{1}{4}(x-1)^2.$$

Hence, our integral is bounded below as follows

$$\int_0^1 [y'^2 + y'(1-x)]dx \geq -\frac{1}{4}\int_0^1 (x-1)^2 dx = -\frac{1}{4}\frac{(x-1)^3}{3}\bigg|_0^1 = -\frac{1}{12}$$

and this bound is achieved when $y' = \frac{1}{2}(x-1)$.

Solving the differential equation $y' = \frac{1}{2}(x-1)$ with $y(0) = 0$ gives the solution $y = \frac{1}{4}(x-1)^2 - \frac{1}{4}$. Therefore, the lower bound $B = -1/12$ of the original integral is achieved when $y = \frac{1}{4}(x-1)^2 - \frac{1}{4}$.

Solution II. (b) First we will show that the function $y(x)$ minimizing the integral must be strictly decreasing. Assume that $y(x)$ is increasing on some interval $[a,b] \subset [0,1]$. Then clearly $y(x) \geq y(a)$, for any $x \in [a,b]$. Hence we obtain that $(y'(x))^2 + y(x) \geq y(a)$. Integrating this over $[a,b]$ we obtain

$$\int_a^b (y'^2 + y)\ dx \geq \int_a^b y(a)\ dx = y(a)(b-a). \tag{1}$$

For a parameter m, define a new function, $g_m(x)$ as follows:

$$g_m(x) = \begin{cases} y(x) & x \leq a \\ y(a) - m(x-a) & a \leq x \leq b\ . \\ y(x) - y(b) + y(a) - m(b-a) & b \leq x \end{cases}$$

We will show that there exists $m > 0$ such that

$$\int_0^1 ((g_m')^2 + g_m)\ dx \leq \int_0^1 (y'^2 + y)\ dx.$$

Since $(g_m')^2 + g_m \leq y'^2 + y$ for every $x \in [0,a]$ and $x \in [b,1]$ then it is sufficient to show that $\int_a^b ((g_m')^2 + g_m)\ dx \leq \int_a^b (y'^2 + y)\ dx$ for some m. Now we have

$$\begin{aligned} \int_a^b ((g_m')^2 + g_m)\ dx &= \int_a^b (m^2 + y(a) - m(x-a))\ dx \\ &= m^2(b-a) - m\frac{(b-a)^2}{2} + y(a)(b-a) \leq y(a)(b-a) \end{aligned}$$

if for example $m = (b-a)/4$. Now our claim follows from (**??**).

Therefore $y(x)$ is a strictly decreasing function on the interval $[0,1]$ and thus it has an inverse function $x(y)$. This yields

$$\int_0^1 y(x)dx = \int_0^{y(1)} [1 - x(y)]dy = \int_0^1 [1-x]y'(x)dx.$$

Therefore

$$\int_0^1 (y'^2 + y)dx = \int_0^1 [y'^2 + (1-x)y']dx.$$

The solution now continues as in Solution I.

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SOLUTIONS II

Problem 4. You come across an old-fashioned paper calendar for the month of May and you see that someone has circled three dates, A, B, and C. You notice that A, B and C are prime numbers and that A, B-1, and B form a Pythagorean triple. While pondering all this, you happen to write down the two-by-two matrix

$$\begin{bmatrix} B & A \\ C & B-1 \end{bmatrix}$$

and you notice that the determinant is 1. That is to say, you notice that $B(B-1) - AC = 1$. What is the product of the three numbers A, B and C?

Solution I. We see that C can be found from A and B using the last equation, so we focus on the possible Pythagorean triples, which can be of two kinds: either $A^2 + (B-1)^2 = B^2$, or else $(B-1)^2 + B^2 = A^2$. We assume the first case, for now, which gives us $A^2 = 2B - 1$, or $B = \frac{A^2+1}{2}$. We can see that A is an odd prime. Since B is a prime of value 31 or less, A must be a prime whose square is 61 or less. This means that A must be 3, 5 or 7. This gives $B = 5$, 13 or 25, respectively, and only the first two are prime. For $A = 3$ and $B = 5$, we find that $C = \frac{19}{3}$, which is not an integer. For $A = 5$ and $B = 13$, we find that $C = 31$, which is a solution with $ABC = 2015$.

Are there any other solutions? If so, they must be ones where $A^2 = B^2 + (B-1)^2$. With $B = 23$ we would have $A^2 = 1013$, which is greater than $(31)^2$. So we only need to check for when B is a prime less than or equal to 19. None of these are such that $B^2 + (B-1)^2$ is the square of a prime.

Solution II. This proof is courtesy a couple of the contestants who noticed that since the numbers A, B and C must be prime integers, it is not necessary to assume that they are between 2 and 31 or to test the individual combinations. One solution did this for both possible forms of the Pythagorean triple, and that solution is essentially reproduced here.

Case 1: Suppose that $A^2 + (B-1)^2 = B^2$. This yields $B = \frac{A^2+1}{2}$. The second condition is that $B(B-1) - AC = 1$, so if we substitute $\frac{A^2+1}{2}$ for B and rearrange, we obtain, $A(A^3 - 4C) = 5$. The only way for 5 to be the product of two such integers with A a positive prime is if $A = 5$ and $A^3 - 4C = 1$, giving us $B = 13$ and $C = 31$. From this, we have the solution $ABC = 2015$.

Case 2: Now suppose that $B^2 + (B-1)^2 = A^2$. This can be put in the form $A^2 - 2B(B-1) = 1$. The other condition can be written $B(B-1) = 1 + AC$, so substituting $1 + AC$ for $B(B-1)$, we obtain $A^2 - 2AC = 3$, or $A(A-2C) = 3$. The only way this can happen is if $A = 3$ and $A - 2C = 1$. But then we must also have $C = 1$, which is not a prime number. (Even if we allow C = 1, there is not even any integer B for which $B(B-1) = AC + 1 = 4$.)

Problem 5. Consider two externally tangent circular discs of radius 1 in the plane. Suppose E is an ellipse that completely encloses the discs and has its major axis on the line joining their centers. What is the smallest possible area of E?

Solution I. Suppose the discs are centered on the x-axis, with their point of tangency at the origin. The boundary of the disc on the right is the circle $(x-1)^2 + y^2 = 1$. (It should be clear that the ellipse is centered at the origin; otherwise, there would be extra space at the left or the right. Then we could slide the discs a bit in that direction, and shrink the ellipse.) The generic equation of such an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a \geq 2$ and $b > 1$. To find the x-coordinate of the intersection of the ellipse and the circle, we eliminate y^2 by a substitution, obtaining $(a^2 - b^2)x^2 - 2a^2x + a^2b^2 = 0$, which gives

$$x = \frac{2a^2 \pm \sqrt{4a^4 - 4a^2b^2(a^2 - b^2)}}{2(a^2 - b^2)}.$$

The ellipse will be tangent to the circle precisely when x is a double root—in other words, when the discriminant $4a^4 - 4a^2b^2(a^2 - b^2)$ is zero. This leads to the equation $a^2 = \frac{b^4}{b^2-1}$.

The area of the ellipse is given by $A = \pi ab$. Minimizing A^2 is a bit easier than minimizing A, so we work with $A^2 = \pi^2 a^2 b^2$ and by substitution we obtain

$$A^2 = \frac{\pi^2 b^6}{b^2 - 1}.$$

Proceeding as usual with $\frac{d}{db}(A^2)$, we find that the minimum occurs at critical point $b = \frac{\sqrt{6}}{2}$, corresponding to $A^2 = \frac{27\pi^2}{4}$, so the minimum area is $A = \frac{3\sqrt{3}\pi}{2}$.

Solution II. The ellipse $Ax^2+By^2=1$, $A>0$, $B>0$ centered at the origin encloses the circles $(x\pm 1)^2+y^2=1$ or $x^2+y^2=2|x|$ if and only if

$$(A-B)x^2+2Bx\le 1 \text{ for all } 0\le x\le 2.$$

In order to maximize the left hand side, the critical point is $x=B/(B-A)$ if $B\ne A$ and this point is between 0 and 2 if and only if $2A<B$.

Case 1. If $A=B$, then the constraint is $B\le 1/4$.
Case 2. If $2A<B$, then the constraint is $B^2\le B-A$ and $A\le 1/4$ since the value at the critical point is

$$-\frac{B^2}{B-A}+\frac{2B^2}{B-A}=\frac{B^2}{B-A}.$$

Case 3. If $2A\ge B$, then the constraint is $A\le 1/4$ (the value at the endpoint $x=2$).

The problem asks to minimize the area $\pi/\sqrt{AB}$ or equivalently to maximize the product AB subject to the constraints above. In cases 1 and 3 the maximum is $AB=1/16$ or $AB=1/8$. In case 2, we need to maximize AB on the region:

$$0<B<2A, \qquad\qquad B^2\le B-A.$$

Notice that the curves $B=2A$ and $B^2=B-A$ meet at the boundary point $(A,B)=(1/4,1/2)$, so that in this region we have $A\le 1/4$. A simple variational argument shows that the maximum of AB must occur on the boundary $A=B-B^2$ for $0<B\le 1/2$. Equivalently, we maximize the function $AB=B^2-B^3$ on the interval $(0,1/2]$ and the maximum occurs at $B=2/3$. So that, the (absolute) minimum area is $\pi/\sqrt{AB}=3\sqrt{3}\pi/2$.

Problem 6. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, such that for any $x\in\mathbb{R}$ and for any $n\in\mathbb{N}$,

$$f(x)\le f\left(x+\frac{1}{n}\right).$$

Show that f is increasing.

Solution I. We must show that $f(x) \leq f(y)$ holds whenever $x < y$. As a tool, we'll use a wonderful fact about the sequence $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Since $\Sigma\frac{1}{n} \to \infty$, while $\frac{1}{n} \to 0$, it follows by an easy exercise that for any $k > 0$ there is a subsequence whose sum converges to k.

Now for any x and y satisfying $x < y$, we set $k = y - x$. Let $\{x_n\}$ be the subsequence mentioned above, whose sum converges to k. By the original given inequality, the sequence

$$f(x), f(x + x_1), f(x + x_1 + x_2), \ldots$$

is increasing. The terms of this sequence have arguments converging to $x+k$, which is y. Hence, by the continuity of f, the sequence converges to $f(y)$. Since the limit of an increasing sequence cannot be smaller than any single term, we have $f(x) \leq f(y)$.

Solution II. It's clear that applying the inequality m times, one gets that

$$f(x) \leq f\left(x + \frac{m}{n}\right)$$

for any $m, n \in \mathbb{N}$. Let $x < y$. Denote $z = y - x > 0$. Any real number can be written as a limit of a sequence of rational numbers. In this case, since $z > 0$, let's write it as a limit of an increasing sequence of positive rational numbers. So $z = \lim\limits_{n\to\infty} z_n$, where $z_n = \frac{p_n}{q_n}$ for some $p_n, q_n \in \mathbb{N}$. Moreover, $f(x) \leq f(x + z_n)$ for every $n > 0$. But $\lim\limits_{n\to\infty}(x + z_n) = x + z = y$, so that $\lim\limits_{n\to\infty} f(x + z_n) = f(y)$ because f is continuous. Since $f(x) \leq f(x + z_n)$ for every $n > 0$, when we take the limit we get $\lim\limits_{n\to\infty} f(x) \leq \lim\limits_{n\to\infty} f(x + z_n)$ and therefore $f(x) \leq f(y)$.

Solution III. (Proof by contradiction.) Suppose that the conclusion is false. Then there are $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Choose an ϵ such that $0 < \epsilon < x_2 - x_1$, and choose a positive integer n such that $\frac{1}{n} < \epsilon$. Now let m be the smallest integer such that $x_1 + \frac{m}{n} > x_2 - \epsilon$. Then, clearly, $x_2 - \epsilon < x_1 + \frac{m}{n} < x_2$. Now, applying the hypothesis m times, we have

$$f(x_2) < f(x_1) \leq f\left(x_1 + \frac{m}{n}\right).$$

But this contradicts the continuity of $f(x)$ at the point x_2 since $x_1 + \frac{m}{n}$ is within ϵ of x_2 and ϵ could be chosen arbitrarily small.