CCSU Regional Math Competition, 2015

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

Problem 1. Suppose the vertices of a hexagon are labeled by the integers 1 through 6, each used just once. For each edge, the absolute difference of the labels at the endpoints is an element of $\{2, 3, 4\}$. Furthermore, the sum of labels at diametrically opposite vertices is never 7. If one edge is chosen at random, what is the probability that the absolute difference of its endpoint labels is 2?

Problem 2. Let R be the operation on 2-by-2 matrices that 'rotates' the entries 90° clockwise. That is, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define $A^R = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$. Find all matrices having real or complex entries and satisfying $A^2 = A^R$.

Problem 3. Let $y = f(x)$ be a function defined on $[0, 1]$. In each of the following cases, find the largest real number B such that the statement

$$\int_0^1 (y'^2 + y)dx \geq B$$

is true for all functions of the type specified:
(a) f is linear, with $f(0) = 0$.
(b) f is continuously differentiable, with $f(0) = 0$.

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Part II

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Problem 4. You come across an old-fashioned paper calendar for the month of May and you see that someone has circled three dates, A, B, and C. You notice that A, B and C are prime numbers and that A, B-1, and B form a Pythagorean triple. While pondering all this, you happen to write down the two-by-two matrix

$$\begin{bmatrix} B & A \\ C & B-1 \end{bmatrix}$$

and you notice that the determinant is 1. That is to say, you notice that $B(B-1) - AC = 1$. What is the product of the three numbers A, B and C?

Problem 5. Consider two externally tangent circular discs of radius 1 in the plane. Suppose E is an ellipse that completely encloses the discs and has its major axis on the line joining their centers. What is the smallest possible area of E?

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N}$,

$$f(x) \leq f\left(x + \frac{1}{n}\right).$$

Show that f is increasing.