CCSU Regional Math Competition, 2013

SOLUTIONS

1. What is the average (arithmetic mean) of the following numbers?

$$2013, 2012, 2012, 2011, 2011, 2011, 2010, 2010, 2010, 2010, \ldots, \underbrace{1, 1, 1, \ldots, 1}_{\text{2013 terms}}$$

Solution 1: Let n be a positive integer. Consider the sequence

$$n, n-1, n-1, n-2, n-2, n-2, \ldots, \underbrace{1, 1, 1, \ldots, 1}_{n \text{ terms}}$$

and denote the average value of the numbers in that sequence by $A(n)$.

We notice first that the number of the elements in our sequence is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

If $n = 1$ then $A(1) = 1$; if $n = 2$ then $A(2) = \frac{4}{3}$; if $n = 3$ then $A(3) = \frac{10}{6} = \frac{5}{3}$; if $n = 4$ then $A(4) = \frac{20}{10} = \frac{6}{3}$; if $n = 5$ then $A(5) = \frac{35}{15} = \frac{7}{3}$. Based on these observations we make the induction hypothesis that $A(n) = \frac{n+2}{3}$ for every positive integer n.

Our hypothesis is clearly true for $n = 1$. Let us suppose that $A(n) = \frac{n+2}{3}$ for some integer $n \geq 1$. We will prove that $A(n+1) = \frac{n+3}{3}$.

Let us compare the following two sequences

$$n, n-1, n-1, n-2, n-2, n-2, \ldots, \underbrace{1, 1, 1, \ldots, 1}_{n \text{ terms}}$$

and

$$n+1, n, n, n-1, n-1, n-1, \ldots, \underbrace{1, 1, 1, \ldots, 1}_{n+1 \text{ terms}}.$$

Since the first sequence has average $A(n) = \frac{n+2}{3}$ and $\frac{n(n+1)}{2}$ elements, the total sum of its elements is

$$\frac{n(n+1)}{2} \cdot \frac{n+2}{3} = \frac{n(n+1)(n+2)}{6}.$$

It is clear that the second sequence has one more 1, one more 2, and so on, one more $n+1$ number than the first sequence. Therefore the total sum of the elements of the second sequence is

$$\frac{n(n+1)(n+2)}{6}+\frac{(n+1)(n+2)}{2}=\frac{(n+1)(n+2)(n+3)}{6}.$$

Since the second sequence has $\frac{(n+1)(n+2)}{2}$ elements, the average value of its elements will be

$$\frac{(n+1)(n+2)(n+3)}{6}\div\frac{(n+1)(n+2)}{2}=\frac{n+3}{3}.$$

Therefore, according to the Principle of Mathematical Induction our claim is true for every positive integer n.

Therefore if $n=2013$, $A(2013)=\frac{2015}{3}$ which is the answer to our question.

Solution 2: The average is given by the following formula for $n=2014$:

$$\frac{\sum_{i=1}^{n-1}(n-i)\cdot i}{\sum_{i=1}^{n-1} i}=n-\frac{\sum_{i=1}^{n-1} i^2}{\sum_{i=1}^{n-1} i}=n-\frac{(n-1)n(2n-1)}{6}\cdot\frac{2}{(n-1)n}$$
$$=n-\frac{2n-1}{3}=\frac{n+1}{3}=\frac{2015}{3}.$$

2. Let a and b be real numbers such that $a<b$. Evaluate

$$\int_a^b \sqrt{(x-a)(b-x)}dx$$

.

Solution 1: We draw a circle with center the point $\frac{b+a}{2}$ and radius $\frac{b-a}{2}$. Let $c=(x,h)$ be any point from that circle. Using similar triangles one can verify that

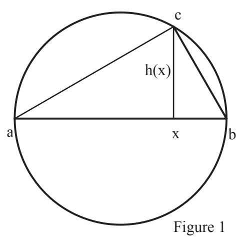

Figure 1

$$h^2=(x-a)(b-x)$$

for every $x\in[a,b]$. Therefore

$$\int_a^b \sqrt{(x-a)(b-x)}dx=\int_a^b h(x)dx$$

which is the area of the upper half circle. Thus,

$$\int_a^b \sqrt{(x-a)(b-x)}dx = \frac{\pi}{2}\left(\frac{b-a}{2}\right)^2 = \frac{\pi(b-a)^2}{8}.$$

Solution 2: First we represent $(x-a)(b-x)$ as a difference of two perfect squares:

$$(x-a)(b-x) = \left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2$$

and thus

$$\int_a^b \sqrt{(x-a)(b-x)}dx = \int_a^b \sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}dx.$$

Then using the substitutions $p = \frac{b-a}{2}$ and $u = x - \frac{a+b}{2}$ we obtain

$$\int_a^b \sqrt{(x-a)(b-x)}dx = \int_{-p}^p \sqrt{p^2-u^2}du.$$

Finally, using the substitution $u = p\sin\theta$, hence $du = p\cos\theta d\theta$, we get consecutively

$$\int_{-p}^p \sqrt{p^2-u^2}du = \int_{-\pi/2}^{\pi/2} p\cos\theta \cdot p\cos\theta d\theta = p^2\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}d\theta$$

$$= p^2\int_{-\pi/2}^{\pi/2} \frac{1}{2}d\theta + \frac{p^2}{4}\int_{-\pi/2}^{\pi/2} \cos 2\theta d2\theta = \frac{\pi p^2}{2} = \frac{\pi(b-a)^2}{8}.$$

3. An open-topped box is constructed from a rectangular sheet R by cutting out a square of side x from each corner and then folding up the four flaps. A calculus student is required to find the value of x for which the volume is maximized. Given that $x = 3$ is the correct answer, and that R has integral length and width, find the largest possible perimeter of R.

Solution: Let R have size a by b where $a \leq b$. Then the volume of the constructed open-top box is

$$V(x) = x(a-2x)(b-2x) = 4x^3 - 2(a+b)x^2 + abx,$$

where $V(x)$ is defined for all positive x such that $x < \frac{a}{2}$. Then $V'(x) = 12x^2 - 4(a+b)x + ab$. Since $V(x)$ has a local maximum at $x = 3$, $V'(3) = 0$. Hence $108 - 12(a+b) + ab = 0$, or $(a-12)(b-12) = 36$. 36 up to symmetry could be written as a product of two integers as follows:

$$36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6.$$

In these cases (a, b) will be equal respectively to

$$(13, 48), (14, 30), (15, 24), (16, 21), (18, 18).$$

Clearly R will have the largest perimeter $P = 122$ when $a = 13$ and $b = 48$. One can verify directly that $V(x)$ actually has a global maximum at $x = 3$ when $a = 13$ and $b = 48$.

4. There are 43 students in a classroom. Each one speaks French or German or Spanish. Each language is spoken by exactly 20 students. Exactly 11 students speak exactly two of these languages. Exactly 5 students speak both German and Spanish. Exactly 33 students speak German or French (or both). What is the probability that 2 students, selected randomly, speak a total of at least 2 of the 3 languages.

Solution: We prepare a Venn diagram as in Figure 2 where with F, G and S are denoted the number of students who speak respectively French, German and Spanish. Then using the text of the problem and the notation introduced on Figure 2 we obtain the following seven equations:

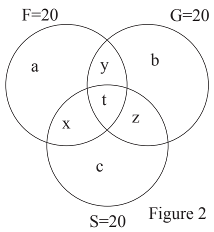

Figure 2

$$a + b + c + x + y + z + t = 43$$
$$a + x + y + t = b + y + z + t = c + x + z + t = 20$$
$$x + y + z = 11, \quad z + t = 5$$
$$a + b + x + y + z + t = 33$$

From the first and the seventh equations we get that $c = 10$ and using the sixth equation we obtain from the fourth equation that $x = 5$. Using the sixth equation again we obtain from the third that $b + y = 15$. Now from the seventh equation we obtain that $a = 8$. Then from the second equation we have $y + t = 7$ and from the fifth we have $y + z = 6$. Adding these two equations and $z + t = 5$ we get $2(y + z + t) = 18$, or $y + z + t = 9$. Therefore $t = 3$, $y = 4$ and $z = 2$; hence $b = 11$. The results are on Figure 3.

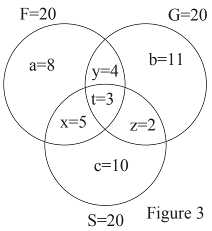

Figure 3

To find the required probability it is sufficient to find the probability of the complementary event: both students to speak only one language which is the same for both of them. This probability is equal to $\frac{8\cdot 7+11\cdot 10+10\cdot 9}{43\cdot 42} = \frac{256}{1806} = \frac{128}{903}$. Therefore the answer to our question is $1 - \frac{128}{903} = \frac{775}{903}$.

5. Does there exist a polynomial function $f(x)$ of degree 4 such that the graph of f''' is tangent to the graph of f at two places?

Solution: Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. Then $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$; $f''(x) = 12ax^2 + 6bx + 2c$; and $f'''(x) = 24ax + 6b$. If $f'''(x)$ is tangent to $f(x)$ at two places then the function

$$g(x) = f(x) - f'''(x) = ax^4 + bx^3 + cx^2 + (d - 24a)x + (e - 6b)$$

must have exactly two double real roots x_1 and x_2. Therefore

$$g(x) = a(x - x_1)^2(x - x_2)^2.$$

If we substitute $a = 1$, $x_1 = 0$ and $x_2 = 1$ we have $g(x) = x^4 - 2x^3 + x^2$ and thus, $b = -2$, $c = 1$, $d - 24 = 0$ and $e - 6(-2) = 0$. Hence $d = 24$ and $e = -12$. Therefore in that case $f(x) = x^4 - 2x^3 + x^2 + 24x - 12$ and one can check directly that $f'''(x)$ is tangent to $f(x)$ exactly at $x = 0$ and $x = 1$. In fact every choice of $a \neq 0$ and $x_1 \neq x_2$ will produce one such polynomial. Therefore the answer of the given question is affirmative.

6. In number theory it is known that for each prime number p and each integer a there is an integer b such that $a^p - a = pb$. Prove or disprove that for each prime number p and each 2×2-matrix with integer entries of the form

$$A = \begin{bmatrix} a & c \\ c & a \end{bmatrix}$$

there is an integer d such that

$$tr(A^p - A) = p \cdot d$$

where $tr(M)$ denotes the sum of the diagonal entries of the matrix M.

Solution 1: We prove that the statement is true. We claim that for any positive integer k

$$A^k = \begin{bmatrix} B & D \\ D & B \end{bmatrix}$$

where B is the sum of all the terms in the binomial expansion of $(a+c)^k$ with even powers of c and D is the sum of the remaining terms in the binomial expansion of $(a+c)^k$. Given that this is the case, we note that

$$tr(A^p - A) = 2B - 2a = (B + D - a - c) + (B - D - a + c)$$

$$= (a+c)^p - (a+c) + (a-c)^p - (a-c) = (b_1 + b_2)p$$

by applying the theorem stated in the problem twice.

In order to prove our claim for the form of A^k we use induction. It is clearly true for $k = 1$, so assume it is true for an arbitrary k. Then

$$A^{k+1} = \begin{bmatrix} a & c \\ c & a \end{bmatrix} \times \begin{bmatrix} B & D \\ D & B \end{bmatrix} = \begin{bmatrix} aB + cD & aD + cB \\ aD + cB & aB + cD \end{bmatrix}.$$

Our claim follows.

Solution 2: We prove that the statement is true. Indeed, the following conjugate of the matrix A is an upper triangular matrix with integer entries

$$S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \qquad SAS^{-1} = \begin{bmatrix} a+c & c \\ 0 & a-c \end{bmatrix} = \begin{bmatrix} a_1 & * \\ 0 & a_2 \end{bmatrix}.$$

Hence, by an inductive calculation we have

$$SA^pS^{-1} = \begin{bmatrix} a_1^p & * \\ 0 & a_2^p \end{bmatrix}.$$

Moreover, the sum of the diagonal entries is invariant under conjugation:

$$M = \begin{bmatrix} m & n \\ r & s \end{bmatrix}, \qquad SMS^{-1} = \begin{bmatrix} m+n & * \\ * & s-n \end{bmatrix}, \qquad tr(SMS^{-1}) = tr(M).$$

The statement now follows from the number theoretical fact mentioned in the problem since

$$tr(A^p - A) = tr(SA^pS^{-1} - SAS^{-1}) = a_1^p - a_1 + a_2^p - a_2.$$