CCSU Regional Math Competition, 2013

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

1. What is the average (arithmetic mean) of the following numbers?

$$2013, 2012, 2012, 2011, 2011, 2011, 2010, 2010, 2010, 2010, \ldots, \underbrace{1, 1, 1, \ldots, 1}_{\text{2013 terms}}$$

2. Let a and b be real numbers such that $a < b$. Evaluate

$$\int_a^b \sqrt{(x-a)(b-x)}dx$$

.

3. An open-topped box is constructed from a rectangular sheet R by cutting out a square of side x from each corner and then folding up the four flaps. A calculus student is required to find the value of x for which the volume is maximized. Given that $x = 3$ is the correct answer, and that R has integral length and width, find the largest possible perimeter of R.

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Part II

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4. There are 43 students in a classroom. Each one speaks French or German or Spanish. Each language is spoken by exactly 20 students. Exactly 11 students speak exactly two of these languages. Exactly 5 students speak both German and Spanish. Exactly 33 students speak German or French (or both). What is the probability that 2 students, selected randomly, speak a total of at least 2 of the 3 languages.

5. Does there exist a polynomial function $f(x)$ of degree 4 such that the graph of f''' is tangent to the graph of f at two places?

6. In number theory it is known that for each prime number p and each integer a there is an integer b such that $a^p - a = pb$. Prove or disprove that for each prime number p and each 2×2-matrix with integer entries of the form

$$A = \begin{bmatrix} a & c \\ c & a \end{bmatrix}$$

there is an integer d such that

$$tr(A^p - A) = p \cdot d$$

where $tr(M)$ denotes the sum of the diagonal entries of the matrix M.