CCSU Regional Math Competition, 2012

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

1. Find all real numbers r, with $r \geq 1$, such that a 1-by-r rectangle R can be cut apart into exactly 3 rectangular pieces, each similar to R.

2. Suppose 4 containers are watched by 4 people as follows: Ann sees containers 1 and 2; Ben sees containers 2 and 3; Cy sees containers 3 and 4; and Dee sees containers 4 and 1. Three balls are tossed into the containers, each ball landing in any of the 4 containers with equal probability. What is the probability that one person sees all 3 balls?

3. For each positive integer $n \geq 2$, define $f(n)$ to be the smallest prime factor of $n(n+1)-1$. For how many values of n not exceeding 2012 does $f(n) = 11$?

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Part II

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4. Take the triangle formed by the centers of the faces that meet at one vertex of a cube and the triangle formed by the centers of the 3 edges meeting at the same vertex. Show that these two triangles are congruent and that one is twice as far from the center of the cube as the other one.

5. Show that

$$\frac{\pi - 2}{\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \frac{x^2 \sin x}{\sqrt{1 + \sin x}} dx \leq \pi - 2.$$

6. Let α and β be positive irrational numbers related by the equation $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Let M be the set of positive integers j such that there exists a positive integer r with $j < r\alpha < j + 1$. Similarly, let N be the set of positive integers k such that there exists a positive integer s with $k < s\beta < k + 1$. Show that $M \cap N = \varnothing$ and that $M \cup N = \mathbb{N}$, the set of all positive integers.

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SOLUTIONS

1. Find all real numbers r, with $r \geq 1$, such that a 1-by-r rectangle R can be cut apart into exactly 3 rectangular pieces, each similar to R.

Solution: We will call a cut of the given rectangle *vertical* if we cut R perpendicularly to the sides equal to r and *horizontal* if the cut is parallel to the sides with length r.

If we cut R horizontally using a full cut, we will obtain two rectangles. Since we need three we can cut only one of these rectangles. Then one of the three rectangles must have dimensions $x \times r$, where $x < 1 \leq r$. In order for this rectangle to be similar to R we must have $x/r = 1/r$, hence $x = 1$, a contradiction. Therefore when we cut R we cannot use full horizontal cuts.

Therefore we must use at least one full vertical cut. Then as above we notice that one of the three rectangles must have dimensions $1 \times x$. The only way for this rectangle to be similar to R is if $1/x = r/1$ or $x = 1/r$. Hence $x \leq 1$. If we want to obtain the other two rectangles by using another vertical cut then we will end up with two rectangles with sides $1 \times y$ and $1 \times z$. Either $y \leq 1$ or $y \geq 1$. In the first case we have $1/x = 1/y$, hence $x = y$. In the second case we have $1/x = y/1$ and since $1/x = r$ we have $r = y$, which is impossible. (For z we have the same possibilities as for y.) Therefore the only such cutting that will produce the required three rectangles is when $x = y = z$. Then we will have three congruent rectangles with sides $1 \times x$. Then $3 \times x = r$ and since $x = 1/r$ we have $r^2 = 3$ or $r = \sqrt{3}$.

If we want to obtain the other two rectangles using horizontal cut then we will obtain two rectangles with dimensions $y \times (r - x)$ and $z \times (r - x)$, where $y + z = 1$. If $y = z$ then $y = z = 1/2$. If $r - x \leq 1/2$ then we must have $1/x = (1/2)/(r - x)$ or $2r = 3x$ and since $x = 1/r$, $r^2 = 3/2$. Therefore $r = \sqrt{3/2}$. If $r - x \geq 1/2$ then we must have $1/x = (r - x)/(1/2)$ or $1 = 2x(r - x)$ and since $x = 1/r$ we have $r^2 = 2$ or $r = \sqrt{2}$.

Now, let $y < z$ (the case $y > z$ will produce the same rectangles). Then we must have $y < r - x < z$ and $y/(r - x) = (r - x)/z$ or $yz = (r - x)^2$. Also we must have $1/x = (r - x)/y$ or $y = x(r - x)$ and thus $y = 1 - 1/r^2$. Therefore $z = 1/r^2$. At the same time $1/x = z/(r - x)$ or $z = (r - x)/x$, hence $z = r^2 - 1$. Therefore $1/r^2 = r^2 - 1$ or $r^4 - r^2 - 1 = 0$. If we substitute $t = r^2$ we obtain $t^2 - t - 1 = 0$ with the only positive solution $t = (1 + \sqrt{5})/2$. Therefore in this case $r = \sqrt{(1 + \sqrt{5})/2}$. To check that this is a solution we have to verify that $yz = (r-x)^2$ or $(1-1/r^2)(1/r^2) = (r-1/r)^2$ or $1/r^2 - 1/r^4 = r^2 - 2 + 1/r^2$. Hence $r^6 - 2r^4 + 1 = 0$. Since $r^2 = t$ we have to verify that $t^3 - 2t + 1 = 0$ and since $t^2 - t - 1 = 0$ we have $t^3 - t^2 - t = 0$ or $t^3 = t^2 + t$. Therefore we have to check that $t^2 + t - 2t^2 + 1 = 0$ or that $-t^2 + t + 1 = 0$, which is clearly true.

Therefore the answer to our question is r could be only $\sqrt{3}$, $\sqrt{2}$, $\sqrt{3/2}$, or $\sqrt{(1+\sqrt{5})/2}$.

2. Suppose 4 containers are watched by 4 people as follows: Ann sees containers 1 and 2; Ben sees containers 2 and 3; Cy sees containers 3 and 4; and Dee sees containers 4 and 1. Three balls are tossed into the containers, each ball landing in any of the 4 containers with equal probability. What is the probability that one person sees all 3 balls?

Solution 1: The first ball could land in any container, thus the probability of that event is 1. Then if we want the first and the second balls to land in containers that are watched by the same person, the second ball must land in the same container or in one of the two containers next to it. (a) The probability the second ball to land in the same container as the first ball is 1/4. (b) The probability the second ball to land in one of the two containers next to it is $2/4 = 1/2$. Finally if we want all three balls to land in containers watched by the same person then in case (a) the third ball must land in the same container or in one of the two containers next to it. The probability of that event is 3/4 and therefore the probability all these three events to happen is $(1) \cdot (1/4) \cdot (3/4) = 3/16$. In case (b) the third ball must land in one of the same two containers where the previous two balls had landed. The probability of that event is 1/2 and therefore the probability all these three events to happen is $(1) \cdot (1/2) \cdot (1/2) = 1/4$. Therefore the answer of the question is $3/16 + 1/4 = 7/16$.

Solution 2: Let A be the event that *Ann* sees all 3 balls and similarly define the events B, C and D. At most two people will see all 3 balls and those two must be side-by-side. This means that the the probability that *someone* sees all 3 balls is the sum of the probabilities of A, B, C and D minus the sum of the probabilities of $A \cap B$, $B \cap C$, $C \cap D$ and $D \cap A$. Since the first four probabilities are equal as well as the last four, the probability we seek is $4\,(P(A) - P(A \cap B))$ in standard notation. For the event A there are 3 ways to obtain 2 balls in cup 1 with 1 ball in cup 2 and likewise 3 ways to obtain 1 ball in cup 1 with 2 balls in cup 2. There is just 1 way for all 3 balls to land in cup 1, and likewise for cup 2. This gives us a total of 8 ways for Ann to see all 3 balls. On the other hand, there is only 1 way for both Ann and Ben to see all 3 balls. The total number of equally likely events is $4 \times 4 \times 4 = 64$, so the probability of someone seeing all 3 balls is $4\left(\frac{8}{64} - \frac{1}{64}\right) = \frac{7}{16}$.

3. For each positive integer $n \geq 2$, define $f(n)$ to be the smallest prime factor of $n(n+1)-1$. For how many values of n not exceeding 2012 does $f(n) = 11$?

Solution: We want to count all numbers of the form $n(n+1)-1 = 11m$ where m is not divisible by 2, 3, 5, or 7. Clearly m is not divisible by 2 since m must be odd. Similarly, by testing the equation $n(n+1)-1 \equiv 0$ modulo 3, 5 and 7, we find that the only possible solutions are when $n \equiv 2 \mod 5$. On the other hand, we find that $n(n+1)-1 \equiv 0 \mod 11$ whenever $n \equiv 3 \mod 11$ or when $n \equiv 7 \mod 11$. This means we are counting all positive integers $n \leq 2012$ of the form $n = 11a+3$ or $n = 11a+7$ such that $n \neq 5b+2$. We find that for $0 \leq a \leq 182$ both forms of n are in range, which gives us $2(183) = 366$ candidates. It remains to subtract those of the form $n = 5b+2$. If $n = 11a+3 = 5b+2$ for $a, b \geq 0$ then $11a = 5b-1$ and $11(a+1) = 5(b+2)$. We set $11(a+1) = 5(b+2) = 55(r+1)$ so that $a, b \geq 0$ whenever $r \geq 0$. Then $n = 5b+2$ coincides with $n = 11a+3$ whenever $n = 55(r+1) - 11 + 3 = 55r + 47 \leq 2012$. This happens for $0 \leq r \leq 35$ or 36 times. Similarly, when $n = 11a+7 = 5b+2$, we find that $11a = 5(b-1) = 55r$ for some $r \geq 0$. This happens for $n = 55r+7$ and $0 \leq r \leq 36$ for a count of 37. We conclude that there are $366 - 36 - 37 = 293$ numbers for which $f(n) = 11$.

4. Take the triangle formed by the centers of the faces that meet at one vertex of a cube and the triangle formed by the centers of the 3 edges meeting at the same vertex. Show that these two triangles are congruent and that one is twice as far from the center of the cube as the other one.

Solution: Denote the cube by $ABCDA'B'C'D'$ and let us denote the midpoints of the edges AA', AB, and AD by M, N and P, and by R, S, and T the centers of the squares $ABB'A'$, $ABCD$, and $ADD'A'$, respectively. ΔANM and $\Delta ABA'$ are similar. Hence $2MN = BA'$. Similarly $2PN = DB$ and $2MP = A'D$. Since $BA' = DB = A'D$ we conclude that ΔMNP is equilateral triangle with sides equal 1/2 of the diagonals of sides of the cube. $\Delta A'TR$ and $\Delta A'DB$ are similar. Hence $2TR = DB$. Similarly $2RS = A'D$ and $2ST = BA'$. Since $BA' = DB = A'D$ we conclude that ΔRST is equilateral triangle with sides equal 1/2 of the diagonals of sides of the cube. Therefore ΔMNP and ΔRST are congruent.

Let O be the center of the cube. Then $OR = AP$ and $OR||AP$; $OS = AM$ and $OS||AM$; and $OT = AN$ and $OT||AN$. Therefore the two pyramids $AMNP$ and $ORST$ are congruent. hence, the distance from O to the plane RST is equal to the distance from A to the plane MNP. But the plane RST is the same as the plane $A'BD$ and therefore the distance from A to the plane MNP is the same as the distance between the planes MNP and RST. Therefore ΔMNP is twice as far from O as ΔRST.

5. Show that

$$\frac{\pi-2}{\sqrt{2}} \le \int_0^{\frac{\pi}{2}} \frac{x^2\sin x}{\sqrt{1+\sin x}}dx \le \pi - 2.$$

Solution: If $x \in [0, \frac{\pi}{2}]$ then $0 \le \sin x \le 1$, hence $1 \le \sqrt{1+\sin x} \le \sqrt{2}$. Therefore

$$\int_0^{\frac{\pi}{2}} \frac{x^2\sin x}{\sqrt{2}}dx \le \int_0^{\frac{\pi}{2}} \frac{x^2\sin x}{\sqrt{1+\sin x}}dx \le \int_0^{\frac{\pi}{2}} x^2 \sin x dx.$$

Also,

$$\int x^2 \sin x dx = -\int x^2 d\cos x = -x^2\cos x + \int 2x\cos x dx$$

$$= -x^2\cos x + \int 2x d\sin x = -x^2\cos x + 2x\sin x - \int 2\sin x dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x.$$

Thus

$$\int_0^{\frac{\pi}{2}} x^2 \sin x dx = (-x^2 \cos x + 2x \sin x + 2 \cos x)\Big|_0^{\frac{\pi}{2}} = \pi - 2.$$

Therefore

$$\frac{\pi - 2}{\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \frac{x^2 \sin x}{\sqrt{1 + \sin x}} dx \leq \pi - 2.$$

6. Let α and β be positive irrational numbers related by the equation $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Let M be the set of positive integers j such that there exists a positive integer r with $j < r\alpha < j + 1$. Similarly, let N be the set of positive integers k such that there exists a positive integer s with $k < s\beta < k + 1$. Show that $M \cap N = \varnothing$ and that $M \cup N = \mathbb{N}$, the set of all positive integers.

Solution: Suppose that $M \cap N \neq \emptyset$. Then there exist positive integers r, s and k such that $k < r\alpha < k+1$ and $k < s\beta < k+1$. Taking the reciprocals and multiplying through by r or by s, we obtain

$$\frac{r}{k+1} < \frac{1}{\alpha} < \frac{r}{k} \qquad \text{and} \qquad \frac{s}{k+1} < \frac{1}{\beta} < \frac{s}{k}.$$

Now adding the two sets of inequalities gives us

$$\frac{r+s}{k+1} < 1 < \frac{r+s}{k}.$$

This says that $r + s$ must lie between k and $k + 1$, which is impossible, so $M \cap N = \varnothing$.

Now suppose that there is some positive integer k such that no integer multiple of either α or β lies between k and $k + 1$. Then there are positive integers r and s such that $r\alpha < k$, $(r+1)\alpha > k+1$, $s\beta < k$ and $(s+1)\beta > k+1$. The first and third inequalities can be re-written as

$$\frac{1}{\alpha} > \frac{r}{k} \qquad \text{and} \qquad \frac{1}{\beta} > \frac{s}{k}$$

Adding the two inequalities tells us that $r + s < k$. By similarly combining the remaining two inequalities we find that $r + s + 2 > k + 1$. Again, we have a contradiction, so $M \cup N = \mathbb{N}$.