**CCSU Regional Math Competition, 2012**

**Part I**

*Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.*

**1.** Find all real numbers $r$, with $r \geq 1$, such that a 1-by-$r$ rectangle $R$ can be cut apart into exactly 3 rectangular pieces, each similar to $R$.

**2.** Suppose 4 containers are watched by 4 people as follows: Ann sees containers 1 and 2; Ben sees containers 2 and 3; Cy sees containers 3 and 4; and Dee sees containers 4 and 1. Three balls are tossed into the containers, each ball landing in any of the 4 containers with equal probability. What is the probability that one person sees all 3 balls?

**3.** For each positive integer $n \geq 2$, define $f(n)$ to be the smallest prime factor of $n(n+1) - 1$. For how many values of $n$ not exceeding 2012 does $f(n) = 11$?

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## Part II

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**4.** Take the triangle formed by the centers of the faces that meet at one vertex of a cube and the triangle formed by the centers of the 3 edges meeting at the same vertex. Show that these two triangles are congruent and that one is twice as far from the center of the cube as the other one.

**5.** Show that

$$\frac{\pi-2}{\sqrt{2}} \le \int_0^{\frac{\pi}{2}} \frac{x^2 \sin x}{\sqrt{1+\sin x}} dx \le \pi - 2.$$

**6.** Let $\alpha$ and $\beta$ be positive irrational numbers related by the equation $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Let $M$ be the set of positive integers $j$ such that there exists a positive integer $r$ with $j < r\alpha < j+1$. Similarly, let $N$ be the set of positive integers $k$ such that there exists a positive integer $s$ with $k < s\beta < k+1$. Show that $M \cap N = \varnothing$ and that $M \cup N = \mathbb{N}$, the set of all positive integers.