CCSU Regional Math Competition, 2011

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

1. A ladder of length L meters is placed vertically against a wall. At a certain moment the base of the ladder begins sliding away from the wall at L meters per minute, continuing until the ladder lies horizontally on the ground. At the same initial moment a mouse at the base of the ladder begins crawling up the ladder, also at L meters per minute. (Thus the mouse begins and ends at the same location at the base of the wall.) At what point in time does the mouse reach its greatest distance from its starting point?

2. Find every positive real number a for which the curves $y = \ln x$ and $y = x^a$ have exactly one point of intersection, or show that no such a exists.

3. In the following table we give names to the six permutations of the three-letter string ABC with subscripts identifying even or odd permutations.

$$\begin{array}{rl@{\qquad\qquad}rl} & \textit{even} & & \textit{odd} \\ a_e = & ABC & a_o = & CBA \\ b_e = & BCA & b_o = & ACB \\ c_e = & CAB & c_o = & BAC \end{array}$$

With the above three-letter strings, or respectively, with their names we construct two infinite strings S and s using the following recursive algorithm. We start S with ABC and s with a_e and then, at each step, we append to S a three-letter string and to s the name of that string in such a way that (1) in s the names of even and odd permutations alternate and (2) the sequence of lower-case letters in s, ignoring the subscripts, is exactly the same as the sequence of the corresponding upper-case letters in S. According to this algorithm the strings s and S begin as follows:

$$\begin{array}{l} s = \quad a_e \quad b_o \quad c_e \quad a_o \quad c_e \quad b_o \quad c_e \quad a_o \quad \ldots \\ S = \overbrace{A\ B\ C}\ \overbrace{A\ C\ B}\ \overbrace{C\ A\ B}\ \overbrace{C\ B\ A}\ \overbrace{C\ A\ B}\ \overbrace{A\ C\ B}\ \overbrace{C\ A\ B}\ \overbrace{C\ B\ A}\ \ldots \end{array}$$

Show that in the string S there is no substring (sequence of consecutive letters of S) of ANY positive length which is immediately followed by the exact same substring.

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Part II

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4. Let $q(x) = ax + b$ be a non-zero polynomial and let $p(x)$ be a polynomial of degree n. Find all functions $f(x) = \frac{p(x)}{q(x)}$ such that f inverse equals f, that is, such that $f^{-1} = f$.

5. Consider quadruples (A, B, C, D) of positive integers having arithmetic mean 2011 and satisfying $A < B < C < D$. Determine the number of distinct quadruples for which the maximum possible value of $\gcd(A, C)$ is attained.

6. Let f be a positive and differentiable function on $(0, \infty)$, and suppose that $\lim_{x\to\infty} \frac{f'(x)}{f(x)} = L$, where $0 < L \le \infty$. Define $f_0(x) = x$ and $f_n(x) = f(f_{n-1}(x))$, for every integer $n \ge 1$. Find $\lim_{x\to\infty} \frac{(f_n(x))^a}{f_{n-1}(x)}$, where $a > 0$ is a real number and $n \ge 1$.