CCSU Regional Math Competition, 2019

SOLUTIONS I

Problem 1. Show that in each year there is a month whose first day is a Sunday and a month whose second day is a Saturday.

Solution. The months of the year have the following number of days mod 7

$$3, 0, 3, 2, 3, 2, 3, 3, 2, 3, 2, 3.$$

In a leap year the second term of the sequence is 1. A closer look of the above sequence shows that if the first day of May is day x of the week ($x = 1$ is a Sunday), then the first day of the 7 consecutive months, beginning with May, will be as follows:

$$x, x + 3, x + 5, x + 1, x + 4, x + 6, x + 2.$$

Since for any x all the remainders mod 7 are listed, then in every year there is a month (between May and November) whose first day is a Sunday. A similar argument shows that there is also a month (between May and November) whose second day is a Saturday.

Problem 2. Find the largest natural number n such that $n + 3$ divides $n^4 + 2019$.

Solution. Using long division we get

$$\frac{n^4 + 2019}{n + 3} = n^3 - 3n^2 + 9n - 27 + \frac{2100}{n + 3}.$$

Therefore the largest n such that $n + 3$ divides $n^4 + 2019$ is $2097 = 2100 - 3$.

Problem 3. Let P be a parabola in the x-y plane having the following properties: the axis of P is parallel to the y-axis, the vertex of P lies on the segment with endpoints $(1, 0)$ and $(0, 1)$, and P passes through the origin. Let R be the bounded region between P and the x-axis. What is the largest possible area of R?

Solution. The segment mentioned in the text of the problem lies on the line $y = 1 - x$. Hence the vertex of P is located at $(t, 1-t)$ for some t in the interval $(0, 1)$. (We note that the endpoint values $t = 0$ and $t = 1$ give degenerate parabolas, and the area of R vanishes in both cases.) The parabola has two x-intercepts: the one at 0 is given, and by symmetry there is another at $2t$. Thus P is the graph of $y = k(x-0)(x-2t)$ for some constant k. The coordinates of the vertex must satisfy this equation, so we have $1 - t = -kt^2$. Solving for k gives $k = \frac{t-1}{t^2}$, and the equation of P thus simplifies to

$$y = \frac{t-1}{t^2}x^2 + \frac{2(1-t)}{t}x.$$

The area of R (which depends on t) can be found by integration:

$$\begin{aligned} A(t) &= \int_0^{2t} \left[\frac{t-1}{t^2}x^2 + \frac{2(1-t)}{t}x\right] dx \\ &= \left(\frac{t-1}{3t^2}x^3 + \frac{1-t}{t}x^2\right)\Bigg|_0^{2t} \\ &= \frac{8t}{3}(t-1) + 4t(1-t) \\ &= \frac{4}{3}(t - t^2). \end{aligned}$$

Now maximizing $A(t)$ is easy: its graph is a parabola that opens downward, so the maximum occurs at the vertex, where $t = \frac{1}{2}$. Hence the largest possible area for R is $A(\frac{1}{2}) = \frac{1}{3}$.

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SOLUTIONS II

Problem 4. Let P be a point inside a regular hexagon with side length 1. Suppose the distance from P to vertex Q is 13/12 and the distance from P to vertex R is 5/12.

a) Show that Q and R are adjacent vertices.

b) Suppose S is the other vertex adjacent to R (i.e. $S \neq Q$). Find the distance from P to S.

Solution.

a) Let A, B, C and D be any four consecutive vertices of a regular hexagon with side length 1. Then the distance between A and B is 1, the distance between A and D is 2, and using the low of cosines, the distance between A and C is $x^2 = 1^2 + 1^2 - 2*1*1*\cos(120°) = 2 + 2*\frac{1}{2} = 3$, or $x = \sqrt{3}$. Since $\frac{5}{12} + \frac{13}{12} = \frac{18}{12} = \frac{3}{2} = 1.5 < \sqrt{3}$, the two vertices Q and R must be adjacent.

b) The side lengths of the triangle ΔPQR are 1, $\frac{5}{12}$, and $\frac{13}{12}$, hence it is a right-angle triangle with right angle at the vertex R. Since the angle measure of $\angle QRS$ is 120°, the angle measure of $\angle PRS$ is 30°. If we denote the length of PS by y, the low of cosines gives us

$$y^2 = (\frac{5}{12})^2 + 1^2 - 2*\frac{5}{12}*1*\cos(30°) = \frac{169 - 60\sqrt{3}}{144}.$$

Therefore the distance from P to S is $\frac{\sqrt{169-60\sqrt{3}}}{12}$.

Problem 5. Let L be a language specified as follows. The alphabet for L is A,B,C,D,E,F. A word in L is any (non-empty) string of letters, provided that the letters occur in alphabetical order and no letter occurs more than once. A sentence in L is any string of words, provided that the words occur in dictionary order, each letter of the alphabet appears in the sentence exactly once, and no two words in the sentence have the same length. (For example, BAD is not a word in L because its letters do not occur in alphabetical order; CF ABDE is not a sentence in L because the two words do not occur in dictionary order.) Find the total number of three-word sentences in L.

Solution. Every sentence has six letters in total. Since the words in a sentence must all have different length, the words in a three-word sentence must have lengths 1, 2, and 3. These lengths can occur in six different possible arrangements. We treat each of these cases separately.

Each sentence must contain the letter A. It must be the first letter of the word containing it, and this word must be the first word of the sentence. This fact will be used in dealing with all six cases. In each case, we consider the number of options available for each word, from first to last.

Case 1: lengths 1-2-3 (example: A BD CEF)
The first word must be A. By similar reasoning, the second word must begin with B. This B can be followed by any of the four unused letters, making four options. This leaves three letters, which must occur in alphabetical order as the third word. The total is $1 \times 4 \times 1 = 4$ sentences of this type.

Case 2: lengths 1-3-2 (example: A BDE CF)
The first word must be A. The second word must begin with B, followed by any two of the four unused letters (in alphabetical order), making $\binom{4}{2} = 6$ options. The remaining two letters determine the third word. The total is $1 \times 6 \times 1 = 6$.

Case 3: lengths 2-1-3 (example: AD B CEF)
The first word must begin with A, followed by any one of the five other letters, forming five options. The second word must consist of the first letter not yet used. The three remaining letters, in alphabetical order, must be the final word. This time we have $5 \times 1 \times 1 = 5$ possible sentences.

Case 4: lengths 2-3-1 (example: AD BCF E)
The first word must begin with A, followed by any one of the five other letters, forming five options. The second word must begin with the first letter not yet used, followed by any two of the remaining three letters (in alphabetical order), making $\binom{3}{2} = 3$ options. The single remaining letter provides the final word. The count is $5 \times 3 \times 1 = 15$.

Case 5: lengths 3-1-2 (example: ABE C DF)
The first word must begin with A, followed by any two of the five other letters (in alphabetical order), forming $\binom{5}{2} = 10$ options. The first letter still available must occur as the second word. The two remaining letters (in alphabetical order) give the third word. This gives $10 \times 1 \times 1 = 10$.

Case 6: lengths 3-2-1 (example: ADE BF C)
The first word must begin with A, followed by any two of the five other letters (in alphabetical order), forming $\binom{5}{2} = 10$ options. The second word must begin with the first letter not yet used, followed by either of the two

remaining letters, forming two options. The last remaining letter must occur as the final word. We have $10 \times 2 \times 1 = 20$ possible sentences of this type.

Totaling these counts gives 60 possible sentences.

Problem 6. Suppose it is snowing at a constant rate (say in inches per hour) and that a snowplow is out plowing snow at a constant rate (in cubic feet per second). During the first hour of plowing the snowplow traveled twice as far as it did during the second hour of plowing. Assume that the snowplow travels in a straight line and is always plowing unplowed snow. For how many hours had it been snowing before the snowplow started plowing?

Solution. Since it is snowing at a constant rate, the height of the snow t hours since it started snowing is

$$h(t) = c_1 t \tag{1}$$

for some $c_1 > 0$. And since the snowplow is plowing at a constant rate, we have

$$v(t)h(t)w = c_2, \tag{2}$$

where $v(t)$ is the velocity of the snowplow at time t, $w > 0$ is the width of the plow, and $c_2 > 0$ is the rate at which snow is being plowed. Substituting (1) into (2) and solving for $v(t)$ gives

$$v(t) = \frac{c_2}{wc_1 t} = \frac{c}{t}, \tag{3}$$

where $c = \dfrac{c_2}{wc_1}$. We obtain the distance traveled by integrating $v(t)$. Suppose the snowplow started plowing p hours after it started snowing. Our goal is to solve for p. Since the snowplow traveled twice as far during the first hour of plowing as it did during the second, we have

$$\int_p^{p+1} v(t)dt = 2\int_{p+1}^{p+2} v(t)dt.$$

Integrating each side gives

$$\int_p^{p+1} v(t)dt = \int_p^{p+1} \frac{c}{t}dt = c[\log(p+1) - \log(p)] = c\log\left(\frac{p+1}{p}\right), \tag{4}$$

and

$$2\int_{p+1}^{p+2} v(t)dt = 2\int_{p+1}^{p+2} \frac{c}{t}dt = 2c[\log(p+2) - \log(p+1)] = c\log\left(\frac{(p+2)^2}{(p+1)^2}\right). \tag{5}$$

Equating (4) and (5) gives

$$\frac{p+1}{p} = \frac{(p+2)^2}{(p+1)^2}.$$

After cross-multiplying we have

$$p^3 + 3p^2 + 3p + 1 = p^3 + 4p^2 + 4p.$$

Combing terms yields the quadratic

$$p^2 + p - 1 = 0,$$

which has positive solution

$$p = \frac{-1+\sqrt{5}}{2} = \phi - 1 \approx 0.618.$$