CCSU Regional Math Competition, 2017

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

Problem 1. Suppose the graph of a quadratic function is concave down and passes through the points $(-1, 1)$ and $(1, 1)$. Find the smallest possible area of the region bounded by the graph and the x-axis.

Problem 2. Solve for the angles A, B, C of a triangle if

$$\cos A + \cos B + \cos C = \frac{3}{2}.$$

Problem 3. For each positive integer n consider the integral

$$I_n = \int_0^1 \frac{dx}{1 + x^{1/n}}.$$

Prove the following three statements:

a) For each n there exists a unique constant $0 < c_n < 1$ such that

$$1 + c_n^{1/n} = 1/I_n.$$

b) The sequence I_n converges to $1/2$.

c) The sequence c_n converges to $1/e$ where e is the Euler number.

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Part II

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Problem 4. Suppose r is a positive real number. Consider the family of circles C_i in the plane, where C_0 has radius 1 and center $(1,1)$ and for each integer $i \geq 1$, the circle C_i has radius r^i, lies in the first quadrant on the right side of C_{i-1}, is externally tangent to C_{i-1}, and is tangent to the x-axis. Let x_i be the x-coordinate of the center of C_i. Find r such that the sequence x_i converges to $7/3$.

Problem 5. Find all triples (x, y, z) of positive real numbers such that

$$x + \frac{2}{y} = 3y \qquad y + \frac{2}{z} = 3z, \qquad z + \frac{2}{x} = 3x.$$

Problem 6. For each positive integer n, let K_n be the graph on n vertices such that every two vertices are connected by an edge which is colored either red or blue. Show that K_n must contain

a) at least two monochromatic triangles if $n = 6$;

b) at least four monochromatic triangles if $n = 7$.

You may assume part a) in part b).