CCSU Regional Math Competition, 2018

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

Problem 1. Let S be the square in a rectangular coordinate plane with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Find a point P inside S such that the vertical line through P and the horizontal line through P split S into four regions whose areas form a (finite) geometric sequence with common ratio π.

Problem 2. Consider a 2×2 matrix A with real entries, whose determinant is $\det A = 2$ and whose trace is $\mathrm{tr}A = 2$. Show that

$$\det(A^2 + xA + I) \geq \frac{1}{2}$$

for all real numbers x, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the trace $\mathrm{tr}A = a + d$ and $\det A = ad - bc$.

Problem 3. Consider the sequence of real numbers defined as

$$a_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}\, dx.$$

Find the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

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Part II

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Problem 4. Find the limit

$$\lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}.$$

Problem 5. We define an operation $\circledcirc$ that interleaves two sequences as follows: For $A = a_1, a_2, \ldots$ and $B = b_1, b_2, \ldots$, let $A \circledcirc B = a_1, b_1, a_2, b_2, \ldots$. Now, setting $C = 1, 0, 1, 0, \ldots$, we define D to be the sequence such that $C \circledcirc D = D$. Find $\displaystyle\sum_{i=1}^{2018} D_i$.

Problem 6. Show that there exists a unique function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following two conditions:

a) $x^3 - (f(x))^3 + xf(x) = 0$ for all $x \in \mathbb{R}$;

b) $f(x) > 0$ for all $x > 0$.