CCSU Regional Math Competition, 2014

SOLUTIONS I

Problem 1. You are floating down the middle of a river. It is 1000 feet wide and flows at 11 feet per second. Suddenly you notice a waterfall 230 feet ahead. Unfortunately, you can only swim 10 feet per second in still water. Can you reach the bank before being swept over the falls?

Solution. Assume that the swimmer's velocity in still water is a constant vector $xi + yj$ with $x^2 + y^2 = 100$. That allows us to write $x = 10\cos\theta$ and $y = 10\sin\theta$ and to look for an angle θ such that at the time T when the swimmer reaches the bank, $xT + 11T \le 230$. Since $yT = 500$, we have

$$T = \frac{500}{y} = \frac{500}{10\sin\theta}.$$

Therefore

$$\frac{10\cos\theta \cdot 500}{10\sin\theta} + \frac{11 \cdot 500}{10\sin\theta} \le 230.$$

Thus, we are looking for an angle $\theta \in [0, \pi]$ such that the following function

$$f(\theta) = 50\cos\theta - 23\sin\theta + 55$$

attains a negative value. Since $f'(\theta) = -50\sin\theta - 23\cos\theta$, the critical point of this function is an angle c where $\tan c = -23/50 = -0.46$. Now using the trigonometric identities

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \quad \text{and} \quad \cos\theta = -\frac{1}{\sqrt{1+\tan^2\theta}}$$

we obtain

$$\sin c = \frac{0.46}{\sqrt{1+0.46^2}} \quad \text{and} \quad \cos c = -\frac{1}{\sqrt{1+0.46^2}}.$$

Then $f(c) < 0$ if and only if

$$55\sqrt{1+0.46^2} < 50 + 23 \cdot 0.46.$$

Computing the squares on both sides we get $3665.09 < 3669.93$ proving that the answer is yes.

Problem 2. We are given 2015 positive integers. We know that if we take away any one of them, the remaining 2014 integers can be partitioned into two sets with the same number of elements and the same sum of elements. Show that all integers must be equal.

Solution. Let $x_1, x_2, ..., x_{2015}$ be the sequence of integers with the given property, say P. Assuming that x_1 is the smallest term, $y_i = x_i - x_1$ is another sequence with the property P but the smallest term is zero. If S is the sum of all y_i, it follows that $S - y_i$ is even for each i. Hence, y_i are all even since $y_1 = 0$. If we divide each y_i by 2 we get another sequence of integers $y_i/2$ with the property P and the smallest term zero. Iterrating the division by 2 we get an infinite descend showing that all $y_i = 0$ and thus, all integers x_i are equal.

Second Solution. The problem says that there is a system of homogeneous linear equations whose $(2n+1) \times (2n+1)$ (coefficient) matrix A has the following three properties:

(1) The diagonal entries are $a_{ii} = 0$.

(2) The off diagonal entries are $a_{ij} = \pm 1$ for $i \neq j$.

(3) The sum of entries in each row equals zero.

Property (3) is invariant under elementary row operations and implies that the rank of the matrix A is at most $2n$. Properties (1) and (2) imply that by adding the first row to each of the other rows we get a submatrix, which is the identity $2n \times 2n$ matrix mod 2. Hence the rank of the matrix A is at least $2n$. We conclude that the reduced echelon form of A has exactly $2n$ entries 1 and satisfies property (3). Thus, A is row equivalent to the following matrix:

$$\begin{bmatrix} 1 & 0 & ... & 0 & -1 \\ 0 & 1 & ... & 0 & -1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 0 \end{bmatrix}.$$

This proves that the $2n+1$ numbers (not necessarily integers) must be equal to each other.

Problem 3. Let P be the parabola $y = x^2$. Let A be any point on P other than the vertex. Let L be the line orthogonal to the tangent line to P at A. Let B be the other point at which L crosses P. Find the smallest possible area of the bounded region lying between P and the segment AB.

Solution. Let $A = (-a, a^2)$ and $B = (b, b^2)$ with $a > 0$ be the two points in the problem. The slope of the tangent line at A is $-2a$ and thus, the slope of the normal line AB is

$$\frac{b^2 - a^2}{b + a} = \frac{1}{2a}, \quad b = a + \frac{1}{2a}.$$

The area under the parabola P between $-a$ and b is

$$A_1 = \int_{-a}^{b} x^2 dx = \frac{b^3 + a^3}{3}$$

while the area under the segment AB between $-a$ and b is

$$A_2 = (b + a) \cdot \frac{b^2 + a^2}{2} = \frac{b^3 + b^2 a + b a^2 + a^3}{2}.$$

The problem asks to minimize the area

$$A = A_2 - A_1 = \frac{(b + a)^3}{6} = \frac{1}{6}\left(2a + \frac{1}{2a}\right)^3.$$

This is the same as minimizing $2a + \frac{1}{2a}$ for $a > 0$. The critical point of this function is $a = 0.5$ and the limits at the ends are both $+\infty$. Hence, the minimal area is $A = \frac{4}{3}$.

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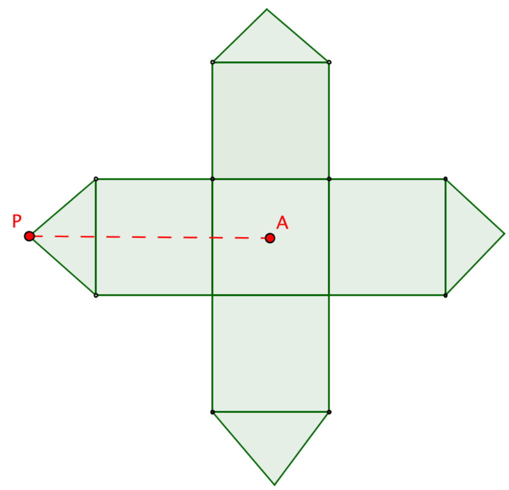
SOLUTIONS II

Problem 4. Show that there is a point A on the surface S of a cube of side 1 that can be joined with any other point on S by a piecewise straight line path contained in S of length at most 2.

Solution. We can build the unit cube by folding the following symmetric planar shape made of five squares and four isosceles right triangles:



By taking A to be the center of symmetry we see that any other point can be connected by a line segment with A. This line segment folds into a broken line on S of length at most $AP = 2$.

Problem 5. Let $x_1, x_2, ..., x_{49}, x_{50}$ be 50 real numbers, not all equal. The mean μ and the standard deviation σ are given by

$$\mu = \frac{1}{50}\sum_{i=1}^{50} x_i, \qquad \sigma = \sqrt{\frac{1}{50}\sum_{i=1}^{50}(x_i - \mu)^2}.$$

The z-score for a particular value x_k is given by $z = \dfrac{x_k - \mu}{\sigma}$. It measures the distance of x_k from the mean in standardized units of σ. Find the largest possible value of the z-score for x_1 and show why it is the largest possible.

Solution. Let $x_1 = b$ and let $x_2 = x_3 = x_4 = ... = x_{50} = a$ where $a < b$. Then for the mean and standard deviation we have

$$\mu = \frac{b + 49a}{50} \qquad \sigma = \sqrt{\frac{(b-\mu)^2 + 49(a-\mu)^2}{50}}$$

Substituting for μ, we find that $b - \mu = \frac{49(b-a)}{50}$ and $a - \mu = \frac{a-b}{50}$. Substituting these into the expression for standard deviation we obtain

$$\sigma = \sqrt{\frac{(49)^2(b-a)^2 + 49(a-b)^2}{(50)^3}} \quad = \quad \frac{\sqrt{49}}{50}(b-a). \tag{1}$$

Then the z-score for x_1 becomes

$$z = \frac{b-\mu}{\sigma} \quad = \quad \sqrt{49} \quad = \quad 7\ .$$

To see why this is the largest possible value, take any other set of values (but not all equal) for the x_i's . Now swap x_1 with whichever x_k is the largest so that x_1 has the highest z-score. We claim that this z-score is less than or equal to 7. Let a be the average of x_2 through x_{50}. If we replace each x_2 through x_{50} by the number a, then the average of all the x_i will remain the same, as will $x_1 - \mu$. On the other hand, the standard deviation will decrease. Since the square of an average of a collection of real numbers is less than or equal to the average of the squares, we have

$$(a-\mu)^2 \leq \sum_{i=2}^{50} \frac{(x_i - \mu)^2}{49}$$

It follows that the revised standard deviation,

$$\sqrt{\frac{49(a-\mu)^2 + (x_1-\mu)^2}{50}}$$

will be less than or equal to the standard deviation (**??**), so that the z-score for x_1 will have increased to 7 (or stayed the same).

Second Solution. If we let $z_k = (x_k - \mu)/\sigma$, then the problem is asking to find the maximum value of the function

$$z_1 = -z_2 - z_3 - ... - z_n = f(z_2, ..., z_n)$$

subject to the constraint

$$g(z_2, z_3, ..., z_n) = (z_2 + z_3 + ... + z_n)^2 + z_2^2 + z_3^2 + ... + z_n^2 = n.$$

The Lagrange multiplier λ is a solution to the system $\nabla f = \lambda \nabla g, g = n$. By the symmetry of the variables, the only solution of the system is obtained when $z_2 = z_3 = ... = z_n = t$ where $(n-1)^2 t^2 + (n-1)t^2 = n$. Hence, the maximum value of z_1 is $\sqrt{n-1}$ and is attained at $x_1 = 1$ and $x_k = 0$ for $k > 1$ when $\mu = 1/n$ and $\sigma = \sqrt{n-1}/n > 0$.

Third Solution. Note that for any distribution of the variables x_1, x_2, ..., x_n there exists a translation such that $x_1 + x_2 + ... + x_n = 0$ or equivalently, $x_1 = -x_2 - x_3 - ... - x_n$. By using this translation, we need to maximize

$$z(x_1) := \frac{x_1\sqrt{n}}{\sqrt{x_1^2 + x_2^2 + ... + x_n^2}} = \sqrt{n} \cdot \left(1 + \frac{x_2^2 + x_3^2 + ... + x_n^2}{(x_2 + x_3 + ... + x_n)^2}\right)^{-\frac{1}{2}}.$$

The following sum of squares is zero if and only if $x_2 = x_3 = ... = x_n$:

$$\sum_{2 \le i < j \le n} (x_i - x_j)^2 = (n-2)\sum_{i=1}^{n} x_i^2 - 2\sum_{2 \le i < j \le n} x_i x_j \ge 0.$$

Equivalently, the following inequality is an equality

$$(n-1)\sum_{i=2}^{n} x_i^2 \ge \left(\sum_{i=2}^{n} x_i\right)^2.$$

if and only if $x_2 = x_3 = ... = x_n$. So, the maximum of $z(x_1)$ is

$$\sqrt{n}\left(1 + \frac{1}{n-1}\right)^{-\frac{1}{2}} = \sqrt{n-1}.$$

Problem 6. Given $C > 0$, find all non-negative continuous functions f defined on $[0, \infty)$, which satisfy the following inequality for all $x \geq 0$

$$f(x) \leq C \cdot \int_0^x f(t)dt.$$

Solution. Since

$$f(x) \leq C \int_0^x f(t)dt$$

and $e^{-Cx} > 0$ we have

$$e^{-Cx} f(x) - e^{-Cx} C \int_0^x f(t)dt \leq 0,$$

or

$$\frac{d}{dx}\left[e^{-Cx} \cdot \int_0^x f(t)dt\right] \leq 0$$

for every $x \geq 0$. This yields by monotony

$$g(x) = e^{-Cx} \cdot \int_0^x f(t)dt \leq g(0) = 0.$$

Since f is continuous and $f \geq 0$, the integral is zero only for $f = 0$.

Second Solution. For a fixed $L > 0$ let M be the maximum of f on the closed interval $[0, L]$. Then

$$f(x) \leq C \cdot \int_0^x M dt = CMx.$$

Hence

$$f(x) \leq C \cdot \int_0^x CMt dt = \frac{(Cx)^2}{2!} \cdot M.$$

Continuing in that way n times, where n is a non-negative integer, we get for all x in $[0, L]$ and $n \geq 0$

$$f(x) \leq \frac{(Cx)^n}{n!} \cdot M \leq \frac{(CL)^n}{n!} \cdot M.$$

Taking the limit as $n \to \infty$ we get $f \leq 0$ and since $f \geq 0$ we get $f = 0$ on $[0, L]$. But $L > 0$ was taken arbitrary and thus, $f = 0$ on $[0, \infty)$.