CCSU Regional Math Competition, 2010

SOLUTIONS

1. Show that a number $n > 1$ is prime if and only if n divides $\binom{n}{k}$ for every integer $k \in [1, n-1]$.

Solution. $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ for every $k \in [0, n]$. If n is prime then n divides $\binom{n}{k}$ for every $k \in [1, n-1]$ since in the numerator there is a factor n and the denominator $k!$ is not divisible by n for every $k < n$. Now let n be a composite number and $k < n$ be a prime number that divides n. Let also m be the largest integer such that k^m divides n. The only factor in the numerator of $\binom{n}{k}$ which is divisible by k is n (in fact n is divisible even by k^m) and its denominator $k!$ is also divisible by k. Therefore $\binom{n}{k}$ is divisible by k^{m-1} but is not divisible by k^m, hence $\binom{n}{k}$ is not divisible by n.

2. a) Prove that the equation

$$x^3 - x - 1 = 0 \qquad (1)$$

has one real and two complex roots.

b) Let a and b be any two of the three roots of (1). Show that $(a-b)^2$ is a root of the equation

$$z^3 - 6z^2 + 9z + 23 = 0. \qquad (2)$$

Solution. a) Let $f(x) = x^3 - x - 1$. Since $f(x)$ is a cubic polynomial it has at least one real root. The derivative of $f(x)$ is $f'(x) = 3x^2 - 1$ and $f'(x) = 0$ for $x = \pm\frac{\sqrt{3}}{3}$. Hence $f(x)$ increases on the intervals $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$ and decreases on the interval $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$. Thus $f(x)$ has a local maximum at $x = -\frac{\sqrt{3}}{3}$, and $f\left(-\frac{\sqrt{3}}{3}\right)$ is negative. Therefore $f(x)$ has only one real root which is in the interval $\left(\frac{\sqrt{3}}{3}, \infty\right)$.

b) Let a, b and c be the three distinct roots of (1). Since $x^3 - x - 1 = (x-a)(x-b)(x-c)$, we conclude that

$$abc = 1, \quad ab + bc + ac = -1 \quad \text{and} \quad a + b + c = 0.$$

Also, $0 = (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac)$ and therefore

$$a^2 + b^2 + c^2 = 2.$$

Since a, b and c are roots of (1) we have

$$a^3 = a + 1,\ b^3 = b + 1,\ c^3 = c + 1$$
$$a^4 = a^2 + a,\ b^4 = b^2 + b \text{ and } c^4 = c^2 + c.$$

Also, $a^2b + ab^2 = ab(a+b) = -abc = -1$, hence

$$a^2b + ab^2 = -1.$$

Now we substitute $z = (a-b)^2$ in (2) and using the above equivalences where appropriate we obtain consequently:

$$(a-b)^6 - 6(a-b)^4 + 9(a-b)^2 + 23$$
$$= (a^3 - 3a^2b + 3ab^2 - b^3)^2 - 6(a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4) + 9a^2 - 18ab + 9b^2 + 23$$
$$= (a + 1 - 3a^2b + 3ab^2 - b - 1)^2 - 6(a^2 + a - 4(a+1)b + 6a^2b^2 - 4a(b+1) + b^2 + b) +$$
$$+9a^2 - 18ab + 9b^2 + 23$$
$$= a^2 - 6a^3b + 9a^4b^2 - 18a^3b^3 + 9a^2b^4 - 6ab^3 + b^2 - 2ab + 12a^2b^2 - 36a^2b^2 - 6a^2 -$$
$$-6b^2 + 48ab + 18a + 18b + 9a^2 - 18ab + 9b^2 + 23$$
$$= -6(a+1)b + 9(a^2 + a)b^2 - 18(a+1)(b+1) + 9a^2(b^2 + b) - 6a(b+1) -$$

$$-24a^2b^2 + 4a^2 + 4b^2 + 28ab + 18a + 18b + 23$$
$$= -6ab - 6b + 9a^2b^2 + 9ab^2 - 18ab - 18a - 18b - 18 + 9a^2b^2 + 9a^2b -$$
$$-6ab - 6a - 24a^2b^2 + 4a^2 + 4b^2 + 28ab + 18a + 18b + 23$$
$$= -6a^2b^2 + 9a^2b + 9ab^2 + 4a^2 + 4b^2 - 2ab - 6a - 6b + 5$$
$$= -6a^2b^2 - 9 + 8 - 4c^2 - 2ab + 6c + 5.$$

Therefore we have to prove that $-6a^2b^2 - 4c^2 - 2ab + 6c + 4 = 0$, or equivalently,

$$-6a^2b^2c^2 - 4c^4 - 2abc^2 + 6c^3 + 4c^2 = 0.$$

We have

$$-6a^2b^2c^2 - 4c^4 - 2abc^2 + 6c^3 + 4c^2$$
$$= -6 - 4(c^2 + c) - 2c + 6(c + 1) + 4c^2 = 0,$$

which completes the proof.

3. Consider the following tree

$$\begin{array}{ccccccc}
 & & & \frac{1}{1} & & & \\
 & & \swarrow & & \searrow & & \\
 & \frac{1}{2} & & & & \frac{2}{1} & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
\frac{1}{3} & & \frac{3}{2} & & \frac{2}{3} & & \frac{3}{1}
\end{array}$$

which continues ad infinitum. Each element of the tree has two children – the rule for generating the children is that $\frac{i}{j}$ has children $\frac{i}{i+j}$ on the left and $\frac{i+j}{j}$ on the right. Prove that every positive rational number appears in this tree exactly once.

Solution. First, we show that each fraction is in reduced form. Clearly this is true at the top of the tree. Suppose $\frac{r}{s}$ is a vertex on the highest level such that r/s is not in reduced form. If $\frac{r}{s}$ is a left child, then its parent is $\frac{r}{s-r}$ is also not in reduced form, contradicting the fact that $\frac{r}{s}$ is the highest such fraction. If $\frac{r}{s}$ is a right child, then its parent is $\frac{r-s}{s}$ leading to the same contradiction.

Next, we show that each fraction occurs at some vertex. 1 occurs. Suppose $\frac{r}{s}$ is a fraction which does not occur, and suppose further that it is the one with smallest denominator, and of those the one with smallest numerator. If $r > s$ then $\frac{r-s}{s}$ cannot occur either, since it would be a parent of $\frac{r}{s}$, but $\frac{r-s}{s}$ has the same denominator as $\frac{r}{s}$ and a smaller numerator, a contradiction. If $r < s$ then $\frac{r}{s-r}$ doesn't occur for the same reason, and it has a smaller denominator than $\frac{r}{s}$, again a contradiction.

Finally, we show that no reduced fraction occurs more than once. First, clearly 1 occurs only once, since otherwise it would be a child of $\frac{r}{s}$, and clearly the rule implies that both children are not equal to 1. Now, suppose some fraction occurs more than once. Let $\frac{r}{s}$ have, among these, the smallest denominator, and then the smallest numerator. If $r < s$ then $\frac{r}{s}$ is a left child of two distinct vertices, each equal to $\frac{r}{s-r}$, which has a smaller denominator, thus leading to a contradiction. If $r > s$, then $\frac{r}{s}$ is a right child of two vertices, each $\frac{r-s}{s}$, which have the same denominator but a smaller numerator, again a contradiction.

4. Find the smallest positive real number x such that the sine of x degrees equals the sine of x radians.

Solution. If we express both angles in degree measure, their sizes are x and $\frac{180x}{\pi}$. For convenience we set $z = \frac{180x}{\pi}$. Now for $x > 0$ we clearly have $z > x$. For very small x the angles represented by x and z will both lie in Quadrant I, so their sine values cannot be equal. As x increases, the z angle will rotate into Quadrant II, while the x angle remains in Quadrant I; it now becomes possible for their sine values to match. In particular, the symmetry that governs this situation is expressed by the identity $\sin\theta = \sin(\pi - \theta)$. For θ in Quadrant I (and $\pi - \theta$ therefore in Quadrant II), it is easy to see by considering reference angles that the identity is not only true, but is in fact the *only* way that equality of sine values can occur. Hence we will have our solution precisely when $z = 180 - x$, or equivalently, $\frac{180x}{\pi} = 180 - x$. This linear equation is easily solved, and we find

$$x = \frac{180\pi}{180 + \pi}.$$

5. Define a triangular array of numbers (somewhat like Pascal's triangle) as follows. For each positive integer n, row n has exactly n terms. Each row begins and ends with 1. The remaining terms are given by the formula

$$a_{n,k} = 1 + a_{n-1,k-1} + a_{n-1,k} - a_{n-2,k-1},$$

where $a_{n,k}$ is the k'th entry of row n. How many times does 2010 appear in the triangle?

Solution. After writing out several rows of the triangle, a simple pattern is observed: reading down along any `diagonal' of the triangle (where the first diagonal is just the left-end sequence of pure 1's, and successive diagonals run parallel to this one), we find an arithmetic sequence.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 5 & & 4 & & 1 & \\
1 & & 5 & & 7 & & 7 & & 5 & & 1
\end{array}$$

Specifically, it appears that $a_{n,k}$ is the $(n - k + 1)$'th term in a sequence that begins with 1 and has uniform step-size $k - 1$. This yields the following formula, which we hope will hold for all terms:

$$a_{n,k} = 1 + (n - k)(k - 1).$$

Proceeding by induction, we let $P(n)$ be the statement that the proposed formula holds for every term in row n. It is trivial to check that $P(1)$ and $P(2)$ are true.

We now consider a single row n with $n \geq 3$, and assume that $P(j)$ is true for each $j < n$. Our formula obviously holds for the first and last terms in the row. For an interior term, we start with the definition and then apply $P(n - 1)$ and $P(n - 2)$ to obtain

$$\begin{aligned}
a_{n,k} &= 1 + a_{n-1,k-1} + a_{n-1,k} - a_{n-2,k-1} \\
&= 1 + [1 + (n-k)(k-2)] + [1 + (n-1-k)(k-1)] - [1 + (n-1-k)(k-2)] \\
&= 1 + 1 + nk - 2n - k^2 + 2k + 1 + nk - n - k + 1 - k^2 + k - 1 - nk + 2n + k - 2 + k^2 - 2k \\
&= 1 + nk - n - k^2 + k \\
&= 1 + (n-k)(k-1),
\end{aligned}$$

as desired. Hence $P(n)$ is true under the stated assumption.

It follows that $P(n)$ holds true for every n. Thus every diagonal sequence is arithmetic, as conjectured.

Now, 2010 will appear in a diagonal sequence precisely when 2009 is a multiple of the step-size. From the prime factorization $2009 = 7 \times 7 \times 41$, we find that the divisors of 2009 are $1, 7, 41, 49, 287$, and 2009. Hence 2010 appears in the triangle exactly six times.

6. Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f''(x)$ exists and is continuous for all x. Suppose that for every $a, b \in \mathbb{R}$ with $b > a$ we have
$$\int_{2a}^{2b} f(x)dx = 4\int_a^b f(x)dx. \qquad (3)$$
Prove that there exists a constant C such that $f(x) = Cx$.

Solution. We begin by using the fundamental theorem of calculus. Fix a and differentiate both sides of (1) with respect to b. We obtain:
$$f(2b) = 2f(b), \text{ for all } b \in \mathbb{R}. \qquad (4)$$
Next, we differentiate (4) twice to obtain:
$$f''(2b) = \frac{1}{2}f''(b), \text{ for all } b \in \mathbb{R}. \qquad (5)$$
Next, from (5) we conclude that
$$f''(b) = \frac{1}{2}f''\left(\frac{b}{2}\right) = \frac{1}{4}f''\left(\frac{b}{4}\right) = \cdots = \frac{1}{2^n}f''\left(\frac{b}{2^n}\right).$$
As $n \to \infty$, we have $\frac{1}{2^n}f''\left(\frac{b}{2^n}\right) \to 0$, and so $f''(b) = 0$ for all $b \in \mathbb{R}$.
Finally by integrating $f''(b) = 0$ twice, we must have $f(b) = Cb + D$ for constants C and D. Recall from (4) that $f(2b) = 2f(b)$, and so
$$C(2b) + D = 2(Cb + D), \text{ hence } D = 0.$$
Therefore there exists C such that $f(b) = Cb$.