CCSU Regional Math Competition, 2009

Solutions

1. Let k be a positive real number. For $x \geq 0$ and $f(t) = |t^2 - k^2|$ find $F(x) = \int_{-x}^{x} f(t)dt$.

Solution. Let $f(t) = |t^2 - k^2|$, where $k > 0$ and $x \geq 0$. Then

$$F(x) = \int_{-x}^{x} f(t)dt = 2\int_{0}^{x} f(t)dt.$$

Case I. $0 \leq x \leq k$.

$F(x) = 2\int_0^x |t^2 - k^2|dt = 2\int_0^x (k^2 - t^2)dt = 2\left(k^2t - \frac{t^3}{3}\right)\Big|_0^x = 2k^2x - \frac{2x^3}{3}$.

Case II. $x > k$.

$F(x) = 2\int_0^x |t^2 - k^2|dt = 2\left[\int_0^k (k^2 - t^2)dt + \int_k^x (t^2 - k^2)dt\right] =$

$2\left[\left(k^2t - \frac{t^3}{3}\right)\Big|_0^k + \left(\frac{t^3}{3} - k^2t\right)\Big|_k^x\right] = 2\left[k^3 - \frac{k^3}{3} + \frac{x^3}{3} - k^2x - \frac{k^3}{3} + k^3\right] =$

$= 2\left[\frac{4}{3}k^3 + \frac{x^3}{3} - k^2x\right] = \frac{8}{3}k^3 + \frac{2}{3}x^3 - 2k^2x.$

Therefore

$$F(x) = \begin{cases} 2k^2x - \frac{2x^3}{3} & , \quad 0 \leq x \leq k \\ \frac{8}{3}k^3 + \frac{2}{3}x^3 - 2k^2x & , \quad x > k \end{cases}.$$

□

2. Find the minimum value of the function $(x - y)^2 + (y - z)^2 + (z - x)^2$, where x, y, z are real numbers with mean 0 (that is, $x + y + z = 0$) and population variance $\frac{1}{3}$ (that is, $x^2 + y^2 + z^2 = 1$).

Solution. $(x-y)^2 + (y-z)^2 + (z-x)^2 = 2(x^2+y^2+z^2) - 2(xy+yz+zx) = 2-2(xy+yz+zx)$. Also, $1-0 = (x^2+y^2+z^2)-(x+y+z)^2 = -2(xy+yz+zx)$. Consequently, $(x-y)^2 + (y-z)^2 + (z-x)^2 = 2+1 = 3$.

Hence, the minimum (and only) value of the given function is 3. □

3. Let n be a positive integer and B_n be an $n \times n$ square board with the standard tiling by n^2 unit squares. Let $C(n)$ be the number of different colorings of B_n that meet the following requirements:

a) each unit square is either black or white;
b) each row contains exactly one black square;
c) each column contains exactly one black square; and
d) the coloring pattern is invariant under a 90° rotation of the board.

Find $C(2009)$ and $C(9002)$.

Solution. We first consider the case where n is even. Let N be the total number of black squares in a valid coloring. Clearly $N = n$, by requirement (b). But by requirement (d), each black square is associated with three others, obtained by successive 90° rotations of the board. All black squares thus fall into disjoint orbits of size 4, so N is a multiple of 4. Since 9002 is not a multiple of 4, there is no valid coloring of this size. Hence $C(9002) = 0$.

For $C(2009)$, we count the number of valid colorings by using an inductive construction procedure.

Starting with $n = 1$, we see immediately that $C(1) = 1$.

Assuming we have a valid $n \times n$ coloring, with n odd, we now show how to construct from it $n+1$ distinct valid colorings of size $(n+4) \times (n+4)$. (The same idea, with only slight modification, could be used for n even.) Starting with the valid $n \times n$ coloring, we first enlarge the board by adding an all-white row or column to each of the 4 sides; our board is now $(n+2) \times (n+2)$. Next, we will insert 2 more rows and 2 more columns, all white, in a carefully controlled way. To do this, we first choose any two adjacent columns; one new column will be inserted between them, and will be called the 'key column.' Because the board has $n+2$ columns, there are $n+1$ available choices for the placement of the key column. Since the number of columns is odd, the chosen placement will assign the key column to either the right or the left half of the board. Simultaneously, we will insert a second column on the other half of the board, exactly mirroring the placement of the key column, as well as two new rows, whose positions will simply be 90° rotations of the positions of the new columns. Finally, after making these four insertions, we color black the uppermost square in the key column, along with its three images under successive 90° rotations. This completes one round of construction. The new board is $(n+4) \times (n+4)$, and it is easy to verify that the new coloring is valid. It is obvious that the $n+1$ available choices all lead to distinct colorings.

The inductive count will be straightforward if we can show that every valid $(n+4) \times (n+4)$ coloring can be obtained by our construction, and that any two colorings obtained from *different* $n \times n$ colorings are themselves different. Both facts become obvious by considering the reversal of the construction procedure, as follows.

Suppose we have a valid $(n+4) \times (n+4)$ coloring, again with n odd. The top row must contain exactly one black square. It cannot lie in the corner position, for then the other corner square would also be black, by the requirement of rotational symmetry. It cannot lie in the center of the row, for then the center column would have a black square at both ends, again by the symmetry requirement. The column in which this black square lies will be deleted, along with its mirror image column which contains the bottom row's black square, as well as the two rows which contain the black squares belonging to the first and last columns. The board will then be $(n+2) \times (n+2)$, with no black squares in the outermost rows and columns, which are also to be deleted. This will leave a $n \times n$ board with a coloring that is easily seen to be valid. Since this reverse construction can always be carried out, the forward construction reaches every valid coloring. Since the reverse construction is deterministic (no choice is required, no ambiguity arises), the forward construction never leads to duplicate results.

It follows that, for n odd, $C(n+4) = (n+1)C(n)$. Since $2009 \equiv 1 \pmod 4$ and $C(1) = 1$, we have $C(2009) = 2006 \times 2002 \times 1998 \times ... \times 6 \times 2 \times 1$. $\square$

4. Let P be a point on the unit circle $x^2 + y^2 = 1$. Let Q be the other endpoint of the chord formed by the line through P and $(0, 2)$, and R be the other endpoint of the chord formed by the line through P and $(0, \frac{1}{2})$. Show that Q and R lie on a horizontal line.

Solution. Let $P(x_0, y_0)$ be any point on the unit circle $x^2 + y^2 = 1$. If P is either $(0, 1)$ or $(0, -1)$ then the claim is obviously true since in those cases Q and R coincide. It is easy to check that both points $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ belong to the unit circle. Let P be the point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$. Then the slope of the line trough P and $(0, 2)$ is $\sqrt{3}$ and the slope of the line trough P and the center of the circle $(0, 0)$ is $-\frac{1}{\sqrt{3}}$. Therefore the line trough P and $(0, 2)$ is perpendicular to the radius of the circle at the point P, hence that line is tangent to the circle. Therefore, in that case, Q coincides with P. Since the second coordinate of P is $\frac{1}{2}$, the line through $P \equiv Q$ and $(0, \frac{1}{2})$ is horizontal. Thus, Q and R lie on a horizontal line. The case when P is the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ is symmetric.

Therefore, in what follows, we can assume that $x_0 \neq 0$ and that $y \neq \frac{1}{2}$.

The equation of the line through P and the point $(0, 2)$ is $y - 2 = \frac{(y_0-2)x}{x_0}$, hence $x = \frac{(y-2)x_0}{y_0-2}$, where clearly $y_0 \neq 2$. Similarly, the equation of the line through P and the point $(0, \frac{1}{2})$ is $y - \frac{1}{2} = \frac{(y_0-\frac{1}{2})x}{x_0}$, hence $x = \frac{(y-\frac{1}{2})x_0}{y_0-\frac{1}{2}}$. Substituting x in the equation of the circle we obtain respectively

$$\left(\frac{(y-2)x_0}{y_0-2}\right)^2 + y^2 = 1 \text{ and } \left(\frac{(y-\frac{1}{2})x_0}{y_0-\frac{1}{2}}\right)^2 + y^2 = 1,$$

or equivalently,

$$(y-2)^2x_0^2 + (y_0-2)^2y^2 = (y_0-2)^2 \text{ and } (y-\tfrac{1}{2})^2x_0^2 + (y_0-\tfrac{1}{2})^2y^2 = (y_0-\tfrac{1}{2})^2.$$

Since $x_0^2 + y_0^2 = 1$ we substitute in the above equations $x_0^2 = 1 - y_0^2$ and after simplification we obtain the equations

$$(-4y_0+5)y^2 - 4(1-y_0^2)y - 5y_0^2 + 4y_0 = 0 \text{ and}$$
$$(-y_0+\tfrac{5}{4})y^2 - (1-y_0^2)y - \tfrac{5}{4}y_0^2 + y_0 = 0.$$

It is easy to see that if we multiply the second equation by 4 we will get the first equation. Thus, both equations are equivalent and therefore they have the same roots. One of those roots is the y-coordinate of P and the other root is the y-coordinate of Q and R, respectively. Hence, Q and R have the same y-coordinates and therefore they lie on the same horizontal line. $\square$

5. Let $S = \mathbb{Z} \times \mathbb{Z}$. Show whether or not there exists an uncountable collection $\mathcal{C}$ of subsets of S which is totally ordered by inclusion (that is, for all $A, B \in \mathcal{C}$, $A \subseteq B$ or $B \subseteq A$.)

Solution. For every $r \in \mathbb{R}$ let $A_r = \{(m,n) | (m,n) \in \mathbb{Z} \times \mathbb{Z}, n \neq 0, \frac{m}{n} < r\}$. Then for $r, t \in \mathbb{R}$, with $r < t$, we have $A_r \subset A_t$ and $A_r \neq A_t$ since between any two different real numbers there is a rational number. Therefore the family $\mathcal{C} = \{A_r | r \in \mathbb{R}\}$ is as required. $\square$