CCSU Regional Math Competition, 2011

SOLUTIONS

1. A ladder of length L meters is placed vertically against a wall. At a certain moment the base of the ladder begins sliding away from the wall at L meters per minute, continuing until the ladder lies horizontally on the ground. At the same initial moment a mouse at the base of the ladder begins crawling up the ladder, also at L meters per minute. (Thus the mouse begins and ends at the same location at the base of the wall.) At what point in time does the mouse reach its greatest distance from its starting point?

Solution: Using the obvious x and y coordinates for the horizontal and vertical components, we see that at time t the base of the ladder is at position tL on the x-axis, the top end of the ladder is at position $L\sqrt{1-t^2}$ on the y-axis, and the fraction of the ladder that the mouse has traversed is $tL/L = t$, leaving $1-t$ as the fraction not yet covered. Hence the coordinates of the mouse are $x = (1-t)tL$ and $y = tL\sqrt{1-t^2}$.

Instead of maximizing the distance D, it is easier to maximize D^2. This is justified since, for $D \geq 0$, D^2 is a strictly increasing function of D. We have

$$D^2 = x^2 + y^2 = 2L^2(t^2 - t^3),$$

and differentiating gives

$$\frac{d}{dt}D^2 = 2L^2 t(2-3t),$$

so there are two critical points, $t = 0$ and $t = 2/3$. A maximum for D^2 (and hence for D) must occur over the closed interval $0 \leq t \leq 1$, and it clearly does not occur at either endpoint. This leaves us with $t = 2/3$, or 40 seconds.

2. Find every positive real number a for which the curves $y = \ln x$ and $y = x^a$ have exactly one point of intersection, or show that no such a exists.

Solution: There is just one such number, $a = 1/e$.

For convenience let $f(x) = \ln(x)$ and $g(x) = x^a$. Clearly $g(x) > f(x)$ at the left extreme (say $0 < x \leq 1$) and again at the right extreme (since $g/f \to \infty$ as $x \to \infty$, by an application of l'Hopital's rule). It follows that the single-intersection situation can occur only at a point where the two curves are tangent, since if they crossed transversely they would have to cross again to satisfy $g > f$ at both ends.

Setting $f' = g'$ and solving, we find $x = (1/a)^{1/a}$. This shows that for any given a there is exactly one point where both curves have the same slope. Hence the tangent intersection will occur precisely when f and g are equal at this special x value. Solving

$$f\left((1/a)^{1/a}\right) = g\left((1/a)^{1/a}\right)$$

yields $a = 1/e$, so the tangent intersection occurs only in this one case, and the intersection occurs at $x = e^e$.

Finally, we must show that with $a = 1/e$ the curves do not intersect at any additional point(s). Suppose to the contrary that there is a second point of intersection, say $x = w$.

Then, by the Mean Value Theorem, there is a point strictly between e^e and w where both curves have the same derivative. But this is impossible: as shown earlier, there is just one point where the derivatives are equal, and this is $x = e^e$.

3. In the following table we give names to the six permutations of the three-letter string ABC with subscripts identifying even or odd permutations.

	even		*odd*
$a_e =$	ABC	$a_o =$	CBA
$b_e =$	BCA	$b_o =$	ACB
$c_e =$	CAB	$c_o =$	BAC

With the above three-letter strings, or respectively, with their names we construct two infinite strings S and s using the following recursive algorithm. We start S with ABC and s with a_e and then, at each step, we append to S a three-letter string and to s the name of that string in such a way that (1) in s the names of even and odd permutations alternate and (2) the sequence of lower-case letters in s, ignoring the subscripts, is exactly the same as the sequence of the corresponding upper-case letters in S. According to this algorithm the strings s and S begin as follows:

$$s = \quad a_e \quad b_o \quad c_e \quad a_o \quad c_e \quad b_o \quad c_e \quad a_o \quad \ldots$$
$$S = \overbrace{A\ B\ C}^{a_e}\ \overbrace{A\ C\ B}^{b_o}\ \overbrace{C\ A\ B}^{c_e}\ \overbrace{C\ B\ A}^{a_o}\ \overbrace{C\ A\ B}^{c_e}\ \overbrace{A\ C\ B}^{b_o}\ \overbrace{C\ A\ B}^{c_e}\ \overbrace{C\ B\ A}^{a_o}\ \ldots$$

Show that in the string S there is no substring (sequence of consecutive letters of S) of ANY positive length which is immediately followed by the exact same substring.

Solution: For convenience, a "triple" will mean a three-letter substring of S which begins in S at a position of the form $3n + 1$. A triple will of necessity be one of our named three-letter strings.

If there are any repeats of any length, there must be a first one. Let f and g be the first such pair of substring of S of any length, where $f = g$, as strings, and g immediately follows f in S. If there are more than one pair beginning at the same position in S, let (f, g) be the shortest such pair and suppose the length of f is N.

Claim 1: $N \neq 1$.

If $N = 1$, then $fg = AA$, $fg = BB$, or $fg = CC$. This cannot happen within a permutation of ABC, so the repeated letter must be the first and last letter of two adjacent triples. But since the odd and even permutations are alternating, we see, by inspection of the table above, that this can only happen if both triples are named by the same lower case letter from $\{a, b, c\}$. But that means that there was a repeat of the form AA, BB or CC at an earlier point in S, which is a contradiction.

Claim 2: $N \neq 2$.

If f is of length 2, then fg is a substring of 4 letters, which must span 2 adjacent triples, with the split between both triples being either 3 and 1, or 2 and 2. In the first case there would be two identical letters in one of the triples, which is impossible. The second case would require that an even triple is followed by an even triple, or an odd by an odd (e. g. if $f = AB$ then the adjacent triples would be CAB followed by ABC, and both are even).

Claim 3: $N \neq 3K$ for any $K \geq 1$.

If f is of length $3K$ and if it is aligned with the triples, then it follows from our construction that there is a repeat of length K in s and therefore there must have been an earlier repeat of length K in S, contradicting our assumption that the pair (f, g) is the first repeat. On the other hand, if f begins in position 2 of a triple, then the first letter of that triple is uniquely determined and must be identical to the last letter of f. It follows that the string of length $3K$ beginning one position earlier than f also immediately follows itself, contradicting our assumption that (f, g) was first. If f begins in position 3 of a triple, then the letter immediately to the right of g is uniquely determined and must be identical to the first letter of g, again giving us a repeat of length $3K$ which is aligned with our triples, leading to an earlier repeat of length K. The beginning of that repeat of length K could conceivably coincide with the beginning of f, but it would contradict our assumption that f had the shortest length of any repeats beginning in that position.

Claim 4: $N \neq 3K + 1$ and $N \neq 3K + 2$ for any $K \geq 1$.

Let F and G be the first three letters of f and g, but not necessarily in that order, and note that $F = G$.

First, assume that F or G is aligned with one of the triples in S. WLOG we can assume that the aligned string is F. Thus F (and therefore G) contains 3 distinct letters; hence WLOG we can assume that $F = ABC = G$. Since $N \neq 3K$, G will not be aligned with any of the triples. Then G will span two triples in one of the following two ways: $(\_AB)(C\_\_)$, or $(\_\_A)(BC\_)$. The different alignments of the two identical substrings f and g completely determines them by proceeding to the right, alternating between the two substrings, and using the requirements imposed by our alternating even and odd permutations of ABC.

For the first case, where the beginning of the unaligned substring looks like $(\_AB)(C\_\_)$, we find that C must always be in first position of each triple in the unaligned substring, and in the third position of each triple in the aligned substring. The other two letters alternate between AB and BA in the remaining positions of each triple. (To verify that take the repeating string that begins with $(\_AB)(C\_\_)$. Since $(\_AB)$ must be a permutation we have $(\_AB) = (CAB) = a_e$. Therefore $(C\_\_)$ must be an odd permutation that begins with C; hence $(C\_\_) = (CBA) = a_o$. Thus $(\_AB)(C\_\_) = (CAB)(CBA)$. Hence the other repeating string with length N must begin with $(ABC)(BA\_)$ and since $(BA\_)$ must be a permutation we have that the second repeating string begins with $(ABC)(BAC)$. Thus the first repeating string with length N must begin with $(\_AB)(CBA)(C\_\_)$ and so on.)

If the aligned substring comes first, this would force an A or B into the last position of the substring. This conflicts with the beginning of the repeat, which looks like $(\_AB)(C\_\_) = (CAB)(CBA)$.

If, on the other hand, the unaligned substring comes first, it must end with either (CAB), or (CBA). In the first case, the beginning 3 letters of the repeat (ABC) would have the same parity. In the latter case, the last letter, and the first letter of the repeating substring would constitute a repeat of length 1.

For the second case, where the beginning of the unaligned substring looks like $(\_\_A)(BC\_)$, we find that A must always be in third position of each triple in the unaligned substring, and in the first position of each triple in the aligned substring. The other two letters alternate between BC and CB in the remaining positions of each permutation. (To verify that take the repeating string that begins with $(\_\_A)(BC\_)$. Since $(BC\_)$ must be a permutation we have $(BC\_) = (BCA) = b_e$. Therefore $(\_\_A)$ must be an odd permutation that ends with A; hence $(\_\_A) = (CBA) = a_o$. Thus $(\_\_A)(BC\_) = (CBA)(BCA)$. Hence the other repeating string with length N must begin with $(ABC)(A\_\_)$ and since $(ABC) = a_e$ is an even permutation $(A\_\_)$ must be an odd permutation that begins with A; thus $(A\_\_) = (ACB) = b_o$. Therefore the second repeating string begins with $(ABC)(ACB)$. Hence the first repeating string with length N must begin with $(\_\_A)(BCA)(CB\_)$ and so on.)

If the unaligned substring comes first, this would force an A into the last position of that substring and then the last letter and the first letter of the repeating substrings would constitute a repeat of length 1.

If, on the other hand, the aligned substring comes first, then that string must end in the triple $(\_\_A)$ where the second repeating string begins. Therefore the first letter of that triple must be an A and at the same time we know that this triple is (CBA)-contradiction.

Now suppose that both of the repeating substrings in S are misaligned with the triples at the beginning end. Since the length of each substring is either $N = 3K + 1$ or $N = 3K + 2$, one of them must start at a position of the form $3M + 2$, and the other at a position of the form $3L$. This means that the *right* end of one of the repeating substrings *is* aligned with a triple. If the end of the first repeating substring is aligned with a triple, then the beginning of the second string will be also aligned, which is a contradiction. Therefore the end of the second substring is always aligned with a triple. Then the mirror image of our previous argument again shows that $N = 3K + 1$ or $N = 3K + 2$ is ruled out.

4. Let $q(x) = ax + b$ be a non-zero polynomial and let $p(x)$ be a polynomial of degree n. Find all functions $f(x) = \frac{p(x)}{q(x)}$ such that f inverse equals f, that is, such that $f^{-1} = f$.

Solution: $f(x) = x$ or $f(x) = \frac{-bx+d}{ax+b}$ with b and d not simultaneously equal zero.

Proof: It is clear that the condition $f^{-1} = f$ is equivalent to the condition $f(f(x)) = x$ for all x in the domain of f.

Claim: $n \leq 1$.

Proof of the claim: If $a = 0$ then f is a polynomial of degree n, and since the degree of $f \circ f$ is n^2 and $f(f(x)) = x$; we conclude that $n = 1$. If $a \neq 0$ and f is not a polynomial, then f has a vertical asymptote, and since the graph of f in the plane $x - y$ is symmetric with respect to the line $y = x$, then f must have a horizontal asymptote; therefore $n \leq 1$.

By the previous claim we have that $f(x) = \frac{cx+d}{ax+b}$. A direct computation shows that

$$f(f(x)) = \frac{c(cx+d) + d(ax+b)}{a(cx+d) + b(ax+b)} = \frac{(c^2+ad)x + (cd+bd)}{(ac+ab)x + (ad+b^2)}$$

Since $f(f(x)) = x$, then $(c^2 + ad)x + (cd + bd) = x((ac + ab)x + (ad + b^2))$ or equivalently,

$$a(c + b)x^2 + (b^2 - c^2)x - d(c + b) = (c + b)(ax^2 + (b - c)x - d) = 0.$$

Therefore, either $c = -b$ and we get that $f(x) = \frac{-bx+d}{ax+b}$ or

$$a = 0, \quad b = c \quad \text{and} \quad d = 0$$

and we get $f(x) = x$.

5. Consider quadruples (A, B, C, D) of positive integers having arithmetic mean 2011 and satisfying $A < B < C < D$. Determine the number of distinct quadruples for which the maximum possible value of $\gcd(A, C)$ is attained.

Solution: Set $G = \gcd(A, C)$. Then $A = pG$ and $C = qG$ for positive integers p and q, with $p < q$. Hence $B = pG + m$ and $D = qG + n$ for positive integers m and n. The constraint on the mean implies that

$$A + B + C + D = 2(p + q)G + (m + n) = 8044.$$

In order for G to be large, $p + q$ and $m + n$ must be small. The smallest possible value for $p + q$ is 3, and in this case the smallest possible value for $m + n$ is 4, since $8044 \equiv 4 \pmod 6$. This corresponds to $G = 1340$. The next smallest possible value for $p + q$ is 4, but this leads to $G \leq 1005$ regardless of the value of $m + n$. Hence the maximum value for G is 1340, with $p = 1$, $q = 2$, and $m + n = 4$. This gives three options: $m = 1, 2, 3$ and $n = 3, 2, 1$, respectively. Therefore exactly three quadruples give the maximum G: $(1340, 1341, 2680, 2683)$, $(1340, 1342, 2680, 2682)$, and $(1340, 1343, 2680, 2681)$.

6. Let $f(x)$ be a positive and differentiable function on $(0, \infty)$, and suppose that $\lim_{x\to\infty} \frac{f'(x)}{f(x)} = L$, where $0 < L \leq \infty$. Define $f_0(x) = x$ and $f_n(x) = f(f_{n-1}(x))$, for every integer $n \geq 1$. Find $\lim_{x\to\infty} \frac{(f_n(x))^a}{f_{n-1}(x)}$, where $a > 0$ is a real number and $n \geq 1$.

Solution: We shall show that $\lim_{x\to\infty} \frac{(f_n(x))^a}{f_{n-1}(x)} = \infty$ for every $a > 1$ and every $n \geq 1$.

Since $\lim_{x\to\infty} \frac{f'(x)}{f(x)} = L > 0$, and $f(x) > 0$ for every $x > 0$, there exists $x_0 > 0$ such that $\frac{f'(x)}{f(x)} > 0$ for every $x > x_0$; hence $f'(x) > 0$ for every $x > x_0$. Therefore $f(x)$, for $x > x_0$, is a strictly increasing function. If $f(x)$ were bounded above then there would have existed a positive number M such that $\lim_{x\to\infty} f(x) = M$, hence $\lim_{x\to\infty} f'(x) = 0$, and therefore $\lim_{x\to\infty} \frac{f'(x)}{f(x)} = 0$. Thus $\lim_{x\to\infty} f(x) = \infty$ and therefore $\lim_{x\to\infty} f_n(x) = \infty$ for every $n \geq 1$. Since $a > 0$, we also have $\lim_{x\to\infty}(f_n(x))^a = \infty$. Therefore to find $\lim_{x\to\infty} \frac{(f_n(x))^a}{f_{n-1}(x)}$ we can use the l'Hopital's rule. For simplicity we let $u = f_{n-1}(x)$. Then $\lim_{x\to\infty} \frac{(f_n(x))^a}{f_{n-1}(x)} = \lim_{u\to\infty} \frac{(f(u))^a}{u} = \lim_{u\to\infty} \frac{a(f(u))^{a-1} f'(u)}{1} = \lim_{u\to\infty} \frac{a(f(u))^a f'(u)}{f(u)} = \infty \cdot L = \infty.$