CCSU Regional Math Competition, 2017

SOLUTIONS I

Problem 1. Suppose the graph of a quadratic function is concave down and passes through the points $(-1, 1)$ and $(1, 1)$. Find the smallest possible area of the region bounded by the graph and the x-axis.

Solution. Since the graph has a vertical axis of symmetry and $(-1, 1)$ and $(1, 1)$ are symmetric about the y-axis, it follows that the vertex of the parabola is of the form $(0, c)$ with $c > 1$. Hence, the quadratic function is of the form $f(x) = c - ax^2$ where $f(1) = c - a = 1$ or $a = c - 1$. Hence, the x-intercepts are the solutions of the equation $c - (c-1)x^2 = 0$ given by

$$x_1 = -\sqrt{\frac{c}{c-1}}, \qquad x_2 = \sqrt{\frac{c}{c-1}}.$$

The area under the graph is given by the integral

$$\begin{aligned} A(c) = \int_{x_1}^{x_2} [c - (c-1)x^2]\, dx &= 2cx_2 - \frac{2}{3} \cdot (c-1)x_2^3 \\ &= \frac{4}{3} \cdot \frac{c^{3/2}}{\sqrt{c-1}}. \end{aligned}$$

It is enough to minimize the following function

$$f(c) = \frac{9}{16} \cdot A(c)^2 = \frac{c^3}{c-1}$$

for $c > 1$. Differentiating using the quotient rule, we have

$$f'(c) = \frac{c^2(2c-3)}{(c-1)^2}$$

and thus, the critical point is $c = 3/2$. Since the limits towards the ends are

$$\lim_{c\to\infty} f(c) = \lim_{c\to 1^+} f(c) = \infty$$

we deduce that $A(3/2) = 2\sqrt{3}$ is the minimal area.

Problem 2. Solve for the angles A, B, C of a triangle if

$$\cos A + \cos B + \cos C = \frac{3}{2}.$$

Solution I. Observe that $C = \pi - (A + B)$ and thus,

$$\cos C = -\cos(A + B).$$

Hence, changing the sum into a product and using the half-angle formula, we can rewrite the equation as

$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - 2\cos^2\frac{A+B}{2} + 1 = \frac{3}{2}.$$

The equation can be further manipulated into a quadratic

$$x^2 - x\cos\frac{A-B}{2} + \frac{1}{4} = 0 \qquad \text{where} \qquad x = \cos\frac{A+B}{2}.$$

The discriminant with respect to x is given by

$$\Delta = \cos^2\frac{A-B}{2} - 1 = -\sin^2\frac{A-B}{2}.$$

To get real roots we must have $A = B$ and by symmetry the initial equation leads to $A = B = C = \pi/3$.

Solution II. One solution is easy to find. Suppose $A = B = C$, so all angles are $\pi/3$. Then the cosines sum to $3/2$, as desired. We claim that this is the only solution. To that end, consider the line tangent to $y = \cos x$ at $x = \pi/3$ given by the graph of the function

$$L(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).$$

We now show that $\cos x \leq L(x)$ for all x in $[0, 2\pi/3]$, with equality holding only at $x = \pi/3$. This is immediate over $[0, \pi/2]$, since $\cos x$ is concave down on that interval. Further, we note that

$$\frac{d}{dx}[L(x) - \cos x] = -\frac{\sqrt{3}}{2} + \sin x,$$

which is positive for x in the open interval $(\pi/3, 2\pi/3)$, hence $L(x) - \cos x$ is increasing on that interval. Since $L(x) - \cos x = 0$ at $x = \pi/3$, we have $L(x) - \cos x > 0$ for all x in $(\pi/3, 2\pi/3]$.

We can now rule out all other triangles. Suppose angles A, B, and C are no greater than $2\pi/3$, and at least one of them is not $\pi/3$. Then, noting that $A + B + C = \pi$, we have

$$\cos A + \cos B + \cos C < L(A) + L(B) + L(C) = \frac{3}{2}.$$

So there is no solution of this type. On the other hand, suppose one angle, say C, is greater than $2\pi/3$. Then $\cos C < -1/2$. Since $\cos x$ is never greater than 1, we have

$$\cos A + \cos B + \cos C < 1 + 1 - \frac{1}{2} = \frac{3}{2}.$$

So there is no solution of this type, either. This proves the claim.

Problem 3. For each positive integer n consider the integral

$$I_n = \int_0^1 \frac{dx}{1 + x^{1/n}}.$$

Prove the following three statements:

a) For each n there exists a unique constant $0 < c_n < 1$ such that

$$1 + c_n^{1/n} = 1/I_n.$$

b) The sequence I_n converges to $1/2$.

c) The sequence c_n converges to $1/e$ where e is the Euler number.

Solution. a) Let f be the function defined for $t \geq 0$ by

$$f(t) = \int_0^t \frac{dx}{1 + x^{1/n}}.$$

According to the Fundamental Theorem of Calculus, f is continuous on the closed interval $[0, 1]$, differentiable on the open interval $(0, 1)$, and

$$f'(t) = \frac{1}{1 + t^{1/n}}.$$

By the Mean Value Theorem, there is c_n in the open interval $(0, 1)$ such that

$$f(1) - f(0) = f'(c_n) \qquad \text{or} \qquad 1 + c_n^{1/n} = \frac{1}{I_n}.$$

Since f' is strictly decreasing, such a number c_n is unique, proving the claim.

b) For every $n > 0$ and $x \in [0, 1]$, we have

$$\frac{1}{2} \leq \frac{1}{1 + x^{1/(n+1)}} \leq \frac{1}{1 + x^{1/n}}.$$

By integrating these inequalities we get $1/2 \leq I_{n+1} \leq I_n$. Hence, the sequence I_n is decreasing and bounded below by $1/2$ and thus, convergent. To calculate its limit, let us fix ϵ such that $0 < \epsilon < 1$ and observe that

$$\frac{1}{1 + x^{1/n}} \leq 1 \text{ for } x \in [0, \epsilon] \qquad \text{and} \qquad \frac{1}{1 + x^{1/n}} \leq \frac{1}{1 + \epsilon^{1/n}} \text{ for } x \in [\epsilon, 1].$$

By integrating these inequalities, we get

$$\int_0^\epsilon \frac{dx}{1 + x^{1/n}} \leq \epsilon \qquad \text{and} \qquad \int_\epsilon^1 \frac{dx}{1 + x^{1/n}} \leq \frac{1 - \epsilon}{1 + \epsilon^{1/n}}.$$

By adding the two inequalities term by term and using the lower bound,

$$\frac{1}{2} \leq I_n \leq \epsilon + \frac{1 - \epsilon}{1 + \epsilon^{1/n}} \qquad \text{so that} \qquad \frac{1}{2} \leq \lim_{n\to\infty} I_n \leq \epsilon + \frac{1 - \epsilon}{2}.$$

By taking $\epsilon \to 0$ we conclude that I_n converges to $1/2$.

c) From a) and b) we know that for every $n > 0$

$$c_n = \left(\frac{1 - I_n}{I_n}\right)^n \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1 - I_n}{I_n} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

Therefore we need to use L'Hôpital's rule to find the limit:

$$\begin{aligned}
\lim_{n\to\infty} \ln c_n &= \lim_{n\to\infty} n \ln\left(\frac{1 - I_n}{I_n}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{1-I_n}{I_n}\right)}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\frac{I_n}{1-I_n} \cdot \left(-\frac{1}{I_n^2} \cdot \frac{\partial I_n}{\partial n}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} n^2 \cdot \frac{1}{I_n(1 - I_n)} \cdot \frac{\partial I_n}{\partial n} \\
&= \lim_{n\to\infty} \frac{1}{I_n(1 - I_n)} \cdot \lim_{n\to\infty} n^2 \frac{\partial I_n}{\partial n} = 4 \lim_{n\to\infty} n^2 \frac{\partial I_n}{\partial n}.
\end{aligned}$$

By differentiating and taking limit under the integral sign, we have

$$\begin{aligned}\lim_{n\to\infty} n^2\frac{\partial I_n}{\partial n} &= \lim_{n\to\infty} n^2\int_0^1 -\frac{x^{1/n}\ln x}{(1+x^{1/n})^2}\cdot\left(-\frac{1}{n^2}\right)dx\\ &= \lim_{n\to\infty}\int_0^1 \frac{x^{1/n}\ln x}{(1+x^{1/n})^2}\,dx = \frac{1}{4}\int_0^1 \ln x\,dx\\ &= \frac{1}{4}\cdot\lim_{\epsilon\to 0^+}(x\ln x - x)|_\epsilon^1 = -\frac{1}{4}.\end{aligned}$$

We conclude that the sequence c_n converges to $1/e$ since $\lim_{n\to\infty}\ln c_n = -1$.

Note. To justify the claim that we can take limit under the integral sign we can use a direct argument. Consider the difference:

$$D_n = \frac{1}{4}\int_0^1 \ln x\,dx - \int_0^1 \frac{x^{1/n}\ln x}{(1+x^{1/n})^2}\,dx = \frac{1}{4}\cdot\int_0^1\left(\frac{1-x^{1/n}}{1+x^{1/n}}\right)^2\ln x\,dx$$

and let us fix ϵ again such that $0<\epsilon<1$. The function f below is decreasing for $t\geq 0$ having negative derivative:

$$f(t) = \frac{1-t}{1+t}, \qquad f'(t) = \frac{-2}{(1+t)^2}.$$

Since $\ln x\leq 0$ for $0<x\leq 1$, we have the following inequalities

$$\begin{aligned}\epsilon\ln\epsilon - \epsilon = \int_0^\epsilon \ln x\,dx &\leq A_n = \int_0^\epsilon\left(\frac{1-x^{1/n}}{1+x^{1/n}}\right)^2\ln x\,dx \leq 0\\ \left(\frac{1-\epsilon^{1/n}}{1+\epsilon^{1/n}}\right)^2\int_\epsilon^1 \ln x\,dx &\leq B_n = \int_\epsilon^1\left(\frac{1-x^{1/n}}{1+x^{1/n}}\right)^2\ln x\,dx \leq 0\end{aligned}$$

for all $n>0$ where A_n, B_n are labels. By taking $n\to\infty$ we deduce that

$$\epsilon\ln\epsilon - \epsilon \leq \lim_{n\to\infty} A_n \leq 0, \qquad \text{and} \qquad \lim_{n\to\infty} B_n = 0.$$

Since $4D_n = A_n + B_n$ we conclude that for every $0<\epsilon<1$ we have

$$\epsilon\ln\epsilon - \epsilon \leq 4\lim_{n\to\infty} D_n \leq 0,$$

and by taking $\epsilon\to 0$ we conclude that D_n converges to 0 proving the claim.

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SOLUTIONS II

Problem 4. Suppose r is a positive real number. Consider the family of circles C_i in the plane, where C_0 has radius 1 and center $(1,1)$ and for each integer $i \geq 1$, the circle C_i has radius r^i, lies in the first quadrant on the right side of C_{i-1}, is externally tangent to C_{i-1}, and is tangent to the x-axis. Let x_i be the x-coordinate of the center of C_i. Find r such that the sequence x_i converges to $7/3$.

Solution I. To begin, we note that $r < 1$. Otherwise, the circles' centers would form an unbounded set rather than settling down around 7/3. Let us examine the segment joining the centers of two adjacent circles, say C_i and C_{i+1}. Its length is $r^i + r^{i+1}$, the sum of the radii. To compute its slope we consider the usual right triangle, where the rise is $r^{i+1} - r^i$ and the run, by the Pythagorean relation, is $2\sqrt{r^{2i+1}}$. Hence the slope is

$$\frac{r^{i+1} - r^i}{2\sqrt{r^{2i+1}}} = \frac{r^i(r-1)}{2r^i\sqrt{r}} = \frac{r-1}{2\sqrt{r}}.$$

Since this value is independent of i, it follows that all the centers lie on one straight line, say L. Since $r < 1$, the slope of L is negative. Since $r^i \to 0$ as $i \to \infty$, it is clear that L crosses the x-axis at 7/3, as the shrinking circles are converging to that point.

Consider now the segment joining $(1,1)$ and $(7/3, 0)$. It lies on L, so its slope can be calculated in two ways:

$$\frac{-1}{4/3} = \frac{r-1}{2\sqrt{r}}.$$

Solving for r, we find the unique positive solution $r = 1/4$.

Solution II. The circles C_i and C_{i-1} are externally tangent if and only if the distance between their centers (x_i, r^i) and (x_{i-1}, r^{i-1}) is $r^i + r^{i-1}$ or

$$(x_i - x_{i-1})^2 + (r^i - r^{i-1})^2 = (r^i + r^{i-1})^2.$$

Solving for $x_i - x_{i-1} > 0$ we get

$$(x_i - x_{i-1})^2 = 4r^i r^{i-1}, \qquad x_i - x_{i-1} = 2r^{(2i-1)/2}.$$

Hence, $x_0 = 1$, $x_1 = 1 + 2r^{1/2}$, and by summation, for each $i \geq 1$ we have

$$x_i = 1 + 2r^{1/2}(1 + r + r^2 + ... + r^{i-1}).$$

If $r \geq 1$ the inequality $x_i \geq 1 + 2ir^{1/2}$ proves that the sequence (x_i) is divergent. Hence, we must have $0 < r < 1$. In this case, by using the fact that the sum of a geometric series with initial term a and ratio $|r| < 1$ is $a/(1-r)$, the limit of the sequence is

$$\lim_{i\to\infty} x_i = 1 + 2r^{1/2}\frac{1}{1-r} = \frac{7}{3}, \qquad \frac{r^{1/2}}{1-r} = \frac{2}{3}, \qquad 2r + 3r^{1/2} - 2 = 0.$$

The last equation is quadratic in $r^{1/2}$ and solving for the positive root, we get $r^{1/2} = 1/2$ or $r = 1/4$.

Problem 5. Find all triples (x, y, z) of positive real numbers such that

$$x + \frac{2}{y} = 3y \qquad y + \frac{2}{z} = 3z, \qquad z + \frac{2}{x} = 3x.$$

Solution I. Notice that $x = y = z = 1$ is a solution. Assume $x > 1$. Then $z = 3x - 2/x > 1$ and thus, $y = 3z - 2/z > 1$. If we add all three equations together we get

$$x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

where the left hand side is > 3 and the right hand side is < 3. Hence, there is no solution with $x > 1$. By symmetry, there is no solution with $0 < x < 1$ either. This concludes that $x = y = z = 1$ is the only solution.

Solution II. Let $f(t) = 3t - 2/t$ be a function defined for $t > 0$. Since the derivative $f'(t) = 3 + 2/t^2$ is positive, we deduce that f is strictly increasing. The system can be reduced to a fixed point problem: $f^3(x) = x$ where f^3 denotes the composition of f with itself 3 times. This problem can be further reduced to the fixed point problem: $f(x) = x$. Indeed, if $f(x) > x$, then by repeated application of f we get $f^3(x) > x$. A similar argument works for $f(x) < x$. The equation $f(x) = x$ is equivalent to $2x = 2/x$ or $x^2 = 1$. Since $x > 0$, the only solution is $x = 1$. Hence, $x = y = z = 1$.

Problem 6. For each positive integer n, let K_n be the graph on n vertices such that every two vertices are connected by an edge which is colored either red or blue. Show that K_n must contain

a) at least two monochromatic triangles if $n = 6$;

b) at least four monochromatic triangles if $n = 7$.

You may assume part a) in part b).

Solution. (a) We first show that K_6 contains one monochromatic triangle. Consider one vertex, say V, of K_6. There are five edges meeting at V; at least three of them must be of the same color, say red, by the Pigeonhole Principle. (If not, there could be at most two red and at most two blue edges, but this would only account for four edges.) Let A, B, and C denote the vertices connected to V by these red edges. If triangle ABC is monochromatic of blue color, then we are done. If not, it must have an edge of red color, which forms a monochromatic triangle with two of the red edges leading to V.

Next we show that K_6 contains a second monochromatic triangle. Suppose ABC is monochromatic of blue color, and let D, E, and F be the other three vertices of K_6. The vertex D has three edges leading to A, B, and C; if two of these edges have blue color, they produce a second monochromatic triangle with one edge of ABC. The same is true of vertices E and F. If no second triangle arises in this way, then each of D, E, and F has at least two edges of red color leading to A, B, or C. Now we consider triangle DEF. If it is monochromatic of blue color, then we are done. If not, it must have an edge of red color. Since the vertices of this edge both have two red edges leading to vertices A, B, or C, two of these edges must share a common vertex by the Pigeonhole Principle. This gives a second monochromatic triangle.

(b) Consider one vertex, say V, of K_7. Ignoring V and the six edges leading to it leaves us with a two-colored K_6, which by part (a) contains two monochromatic triangles, say T and U.

Case 1. The two monochromatic triangles share a vertex, say Z. By ignoring Z and invoking part (a) for the K_6 graph not containing Z, we get two additional monochromatic triangles, for a total of four.

Case 2. The two monochromatic triangles do not share a vertex. Then they must use all six vertices of K_7 other than V. We pick one of these six, say a vertex W of triangle T, and again invoke part (a) for the K_6 graph not containing W but containing triangle U. This gives us at least one monochromatic triangle other than T and U, for a total of three.

Since only seven vertices are available in K_7 and we have at least three monochromatic triangles, two of these triangles must share a vertex by the Pigeonhole Principle. By Case 1, we conclude that K_7 has at least four monochromatic triangles.