CCSU Regional Math Competition, 2010

PART I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

1. Show that an integer $n > 1$ is prime if and only if n divides $\binom{n}{k}$ for every integer $k \in [1, n-1]$. (Note: $\binom{n}{k}$ is defined to be $\frac{n!}{k!(n-k)!}$.)

2. a) Prove that the equation

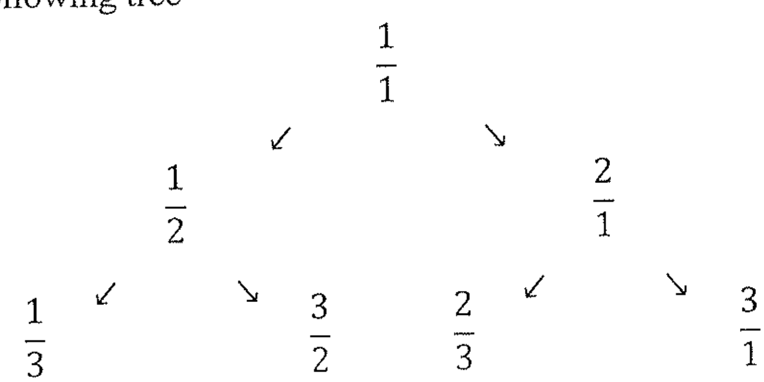
$$x^3 - x - 1 = 0 \qquad (1)$$

has one real and two complex roots.

b) Let a and b be any two of the three roots of (1). Show that $(a-b)^2$ is a root of the equation

$$z^3 - 6z^2 + 9z + 23 = 0.$$

3. Consider the following tree



which continues ad infinitum. Each element of the tree has two children – the rule for generating the children is that $\frac{i}{j}$ has children $\frac{i}{i+j}$ on the left and $\frac{i+j}{j}$ on the right. Prove that every positive rational number appears in this tree exactly once.

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PART II

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4. Find the smallest positive real number x such that the sine of x degrees equals the sine of x radians.

5. Define a triangular array of numbers (somewhat like Pascal's triangle) as follows. For each positive integer n, row n has exactly n terms. Each row begins and ends with 1. The remaining terms are given by the formula

$$a_{n,k} = 1 + a_{n-1,k-1} + a_{n-1,k} - a_{n-2,k-1}, \text{ for } n \geq 3,\ 2 \leq k \leq n-1.$$

where $a_{n,k}$ is the kth entry of row n. How many times does 2010 appear in the triangle?

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f''(x)$ exists and is continuous for all x. Suppose that for every $a, b \in \mathbb{R}$ with $b > a$ we have

$$\int_{2a}^{2b} f(x)dx = 4\int_a^b f(x)dx.$$

Prove that there exists a constant C such that $f(x) = Cx$.