CCSU Regional Math Competition, 2009

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

1. Let k be a positive real number. For $x \geq 0$ and $f(t) = |t^2 - k^2|$ find $F(x) = \int_{-x}^{x} f(t)dt$.

2. Find the minimum value of the function $(x-y)^2 + (y-z)^2 + (z-x)^2$, where x, y, z are real numbers with mean 0 (that is, $x + y + z = 0$) and population variance $\frac{1}{3}$ (that is, $x^2 + y^2 + z^2 = 1$).

3. Let n be a positive integer and B_n be an $n \times n$ square board with the standard tiling by n^2 unit squares. Let $C(n)$ be the number of different colorings of B_n that meet the following requirements:

   a) each unit square is either black or white;
   b) each row contains exactly one black square;
   c) each column contains exactly one black square; and
   d) the coloring pattern is invariant under a 90° rotation of the board.

   Find $C(2009)$ and $C(9002)$.

4. Let P be a point on the unit circle $x^2 + y^2 = 1$. Let Q be the other endpoint of the chord formed by the line through P and $(0, 2)$, and R be the other endpoint of the chord formed by the line through P and $(0, \frac{1}{2})$. Show that Q and R lie on a horizontal line.

5. Let $S = \mathbb{Z} \times \mathbb{Z}$. Show whether or not there exists an uncountable collection $\mathcal{C}$ of subsets of S which is totally ordered by inclusion (that is, for all $A, B \in \mathcal{C}$, $A \subseteq B$ or $B \subseteq A$.)