## CCSU Regional Math Competition, 2016

## SOLUTIONS I

**Problem 1.** For each real number $t \in [-1, 1]$ let $P_t$ be the parabola in the $xy$-plane that has axis parallel to the $y$-axis, passes through the points $(0, 0)$ and $(4, t)$, and has a tangent line with a slope $t - 1$ at the point $(4, t)$. Find the smallest possible $y$-coordinate for the vertex of $P_t$.

**Solution.** For $t \in [-1, 1]$, let $p_t(x) = ax^2 + bx + c$ be the quadratic function whose graph is the parabola $P_t$. Since $p_t(0) = 0$ and $p_t(4) = t$, we have $c = 0$ and $16a + 4b = t$. From $p'_t(x) = 2ax + b$ and $p'_t(4) = t - 1$, we have $8a + b = t - 1$. Hence $a = \frac{1}{16}(3t - 4)$ and $b = \frac{1}{2}(-t + 2)$. Thus,

$$p_t(x) = \frac{3t - 4}{16}x^2 + \frac{-t + 2}{2}x.$$

The $y$-coordinate of the vertex of $P_t$ will be given by the formula

$$p_t\left(\frac{-b}{2a}\right) = p_t\left(\frac{4t - 8}{3t - 4}\right) = \frac{(t - 2)^2}{4 - 3t}.$$

In order to find the lowest possible $y$-coordinate of the vertex of $P_t$, we need to find the minimum value of the function $f(t) = \frac{(t-2)^2}{4-3t}$, when $t \in [-1, 1]$. The derivative of $f(t)$ with respect to $t$ is given by the formula

$$f'(t) = \frac{-3t^2 + 8t - 4}{(4 - 3t)^2}$$

and hence, $f'(t) = 0$ when $t = 2$ or $t = \frac{2}{3}$. Since $2 \notin [-1, 1]$ and

$$f(1) = 1, \qquad f(-1) = \frac{9}{7}, \qquad f\left(\frac{2}{3}\right) = \frac{8}{9},$$

we conclude that the lowest possible $y$-coordinate of the vertex of $P_t$ is $\frac{8}{9}$.

**Problem 2.** Inside a square of side 2 there are 7 polygons each of area 1. Show that there are 2 polygons that overlap over a region of area at least $\frac{1}{7}$.

**Solution.** Let $P_1$, $P_2$, ..., $P_7$ be the 7 polygons and denote by $|R|$ the area of the region $R$. Assume that any 2 of them share an area $< 1/7$. Then

$$|P_1 \cup P_2| = |P_1| + |P_2| - |P_1 \cap P_2| > 2 - 1/7 = 13/7,$$
$$|P_1 \cup P_2 \cup P_3| = |P_1 \cup P_2| + |P_3| - |(P_1 \cup P_2) \cap P_3| > 13/7 + 1 - 2/7 = 18/7.$$

Following the pattern, the union of all polygons will cover an area more than

$$\begin{aligned} &18/7 + 1 - 3/7 = 22/7, \\ &22/7 + 1 - 4/7 = 25/7, \\ &25/7 + 1 - 5/7 = 27/7, \\ &27/7 + 1 - 6/7 = 28/7 = 4. \end{aligned}$$

But the area of the square is 4 and the polygons cannot cover an area more than 4. Hence, our assumption is false and the opposite statement is true.

**Problem 3.** Consider two matrices $A$ $(m \times n)$ and $B$ $(n \times m)$ with real entries, such that $m \geq n \geq 2$. Assume there exist an integer $k \geq 1$ and real numbers $a_0, a_1, ..., a_k$ such that

$$a_k(AB)^k + a_{k-1}(AB)^{k-1} + \cdots + a_2(AB)^2 + a_1(AB) + a_0 I_m = O_m,$$
$$a_k(BA)^k + a_{k-1}(BA)^{k-1} + \cdots + a_2(BA)^2 + a_1(BA) + a_0 I_n \neq O_n,$$

where $I_m$, $I_n$ are the identity matrices and $O_m$, $O_n$ are the zero matrices of the corresponding sizes. Prove that $a_0 = 0$.

**Solution I.** Assume $a_0 \neq 0$. Divide the first identity by $a_0$ and factor $AB$:

$$AB\left(-\frac{a_k}{a_0}(AB)^{k-1} - \frac{a_{k-1}}{a_0}(AB)^{k-2} - \cdots - \frac{a_2}{a_0}(AB) - \frac{a_1}{a_0}I_m\right) = I_m.$$

It follows from this that $AB$ is invertible, and since it is an $m \times m$ matrix, it has to have rank $m$. So $\operatorname{rank}(AB) = m \geq n$. However, using the rank inequality, we know that $m = \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\} \leq n$, so we have that $m = n$ and $A, B, AB$, and $BA$ are all invertible square matrices.

After multiplying the identity

$$a_k(AB)^k + a_{k-1}(AB)^{k-1} + \cdots + a_2(AB)^2 + a_1(AB) + a_0 I_n = O_n$$

to the left with $B$ and to the right with $A$ we obtain

$$a_k(BA)^{k+1} + a_{k-1}(BA)^k + \cdots + a_2(BA)^3 + a_1(BA)^2 + a_0(BA) = O_n.$$

Finally, after multiplying the whole expression by $(BA)^{-1}$ we obtain that

$$a_k(BA)^k + a_{k-1}(BA)^{k-1} + \cdots + a_2(BA)^2 + a_1(BA) + a_0 I_n = O_n,$$

which contradicts the hypothesis. Therefore $a_0 = 0$.

**Solution II.** Observe that $(AB)^j = A(BA)^{j-1}B$ for any $j \geq 1$ and thus, the first equation gives

$$A\left(a_k(BA)^{k-1} + a_{k-1}(BA)^{k-2} + \cdots + a_1 I_n\right)B = -a_0 I_m$$

and if we denote $a_k(BA)^{k-1}+a_{k-1}(BA)^{k-2}+\cdots+a_1 I_n$ by $L$, the two equations can be written as

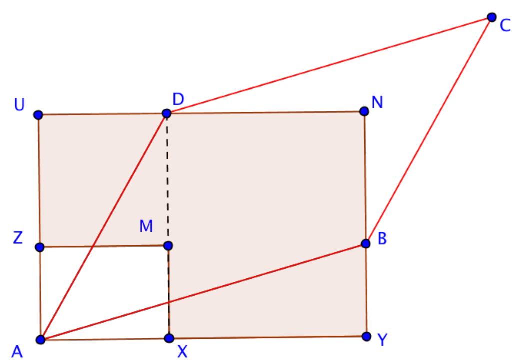
$$ALB = -a_0 I_m, \quad LBA \neq -a_0 I_n.$$

If $a_0 \neq 0$, it follows that $A$ is the matrix of a surjective linear map from $\mathbb{R}^n$ onto $\mathbb{R}^m$ and since $m \geq n$, we deduce that $m = n$ and $A$ is invertible. In particular, $A$ commutes with $LB$ and thus, the second condition contradicts the first one. This proves that $a_0 = 0$.

**CCSU Regional Math Competition, 2016**

**SOLUTIONS II**

**Problem 4.** Show that the area of the region $MXYNUZ$ equals the area of the parallelogram $ABCD$ where the lines $\overleftrightarrow{AY}$, $\overleftrightarrow{BZ}$, $\overleftrightarrow{NU}$ are perpendicular to the line $\overleftrightarrow{AU}$ and the lines $\overleftrightarrow{DX}$, $\overleftrightarrow{NY}$ are perpendicular to the line $\overleftrightarrow{AY}$. The segments $\overline{DX}$, $\overline{BZ}$ meet at the point $M$ and their endpoints are on the sides of the polygon $AYNU$ as in the figure.




**Solution I.** The components of the vector $\overrightarrow{AB}$ are $AY$, $AZ$ and the components of the vector $\overrightarrow{AD}$ are $AX$, $AU$. Hence, the area of the parallelogram $ABCD$ is the determinant

$$\det \begin{pmatrix} AY & AX \\ AZ & AU \end{pmatrix} = AY \times AU - AX \times AZ.$$

The area of the region $MXYNUZ$ is the difference between areas of two rectangles: $AYNU$ and $AXMZ$, and this difference agrees with the determinant above, proving the statement.

**Solution II.** If $R$ is a region in the plane, then by $|R|$ we will denote its area. Since two triangles with the same base and height have the same area, $|ABY| = |AMY|$ and $|ADU| = |AMU|$. Since each diagonal splits the rectangle into two triangles of the same area, we conclude that

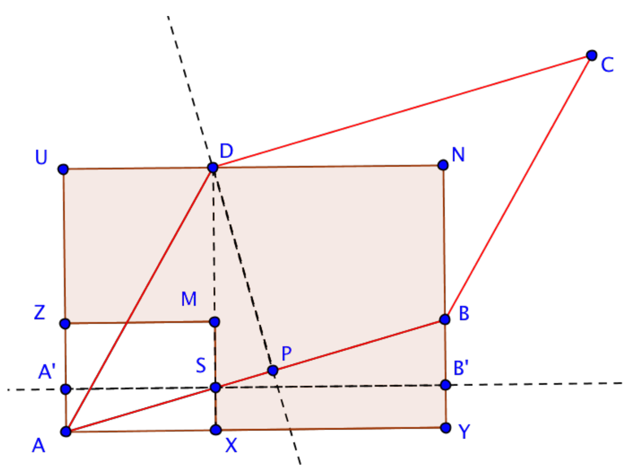
$$\begin{aligned}
|MXYNUZ| &= 2|MUZ| + 2|MXY| + 2|BND|,\\
|MXAZ| &= |MAX| + |MAZ|,\\
|BAY| &= |MAX| + |MXY|,\\
|ADU| &= |MAZ| + |MUZ|,\\
|ABD| &= |AYNU| - |ADU| - |BAY| - |BND|.
\end{aligned}$$

Combining these equations we conclude that

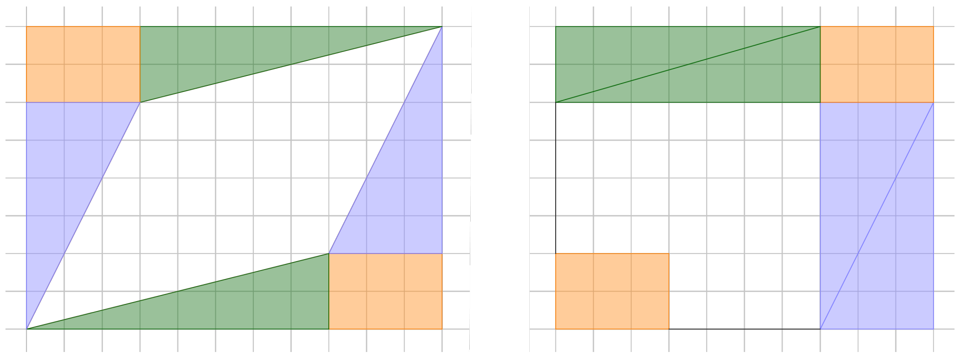
$$\begin{aligned}
|ABD| &= |AYNU| - |MAZ| - |MUZ| - |MAX| - |MXY| - |BND|\\
&= |AYNU| - |MXAZ| - \frac{1}{2}|MXYNUZ| = \frac{1}{2}|MXYNUZ|.
\end{aligned}$$

Since $|ABCD| = 2|ABD|$, our claim is proved.

**Solution III.** Denote the intersection point between the segments $\overline{MX}$ and $\overline{AB}$ by $S$ and the intersection point between the line $\overleftrightarrow{AB}$ and the perpendicular from $D$ to $\overleftrightarrow{AB}$ by $P$ (see the figure below). Let also the line through $S$ parallel to $\overleftrightarrow{AY}$ intersect $\overline{AZ}$ at $A'$ and $\overline{YB}$ at $B'$. Notice that $\triangle AXS$ and $\triangle BMS$ are similar and therefore $AX \times MS = XS \times MB$ and since $AX = ZM$ and $MB = XY$, we have $MZ \times MS = XS \times XY$. Therefore $ZMSA'$ and $XYB'S$ have the same areas, thus $MXYNUZ$ and $A'B'NU$ have the same areas. Now notice that $\triangle AYB$ and $\triangle DPS$ are similar and therefore $AY \times DS = AB \times DP$. Since $AY = A'B'$, we have that $A'B'ND$ and $ABCD$ have the same areas. The proof is completed.



**Solution IV.** Solution without words.



**Problem 5.** Compute the integral

$$\int_0^{\pi/4} \ln(1+\tan x)dx.$$

**Solution.** Using the identities

$$1+\tan x = \frac{\sin x + \cos x}{\cos x},$$
$$\ln\left(\frac{\sin x + \cos x}{\cos x}\right) = \ln(\sin x + \cos x) - \ln\cos x,$$
$$\sin x + \cos x = \sqrt{2}\sin\left(\frac{\pi}{4}+x\right),$$

we separate the given integral into three parts as follows:

$$\begin{aligned}
&\int_0^{\pi/4} \ln(1+\tan x)dx \\
&= \int_0^{\pi/4} \ln(\sin x + \cos x)dx - \int_0^{\pi/4} \ln\cos x \ dx \\
&= \int_0^{\pi/4} \ln\left(\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right) dx - \int_0^{\pi/4} \ln\cos x \ dx \\
&= \int_0^{\pi/4} \ln\sqrt{2} \ dx + \int_0^{\pi/4} \ln\left(\sin\left(\frac{\pi}{4}+x\right)\right) dx - \int_0^{\pi/4} \ln\cos x \ dx \\
&= \frac{\pi\ln\sqrt{2}}{4} + \int_0^{\pi/4} \ln\left(\sin\left(\frac{\pi}{4}+x\right)\right) dx - \int_0^{\pi/4} \ln\cos x \ dx.
\end{aligned}$$

The change of variables $x = \frac{\pi}{4} - t$ shows that the last two integrals are equal and therefore they cancel out. Thus, the answer is $\frac{\pi \ln \sqrt{2}}{4} = \frac{\pi \ln 2}{8}$.

**Problem 6.** Let $f$ be the function defined recursively by $f(0) = 1$ and $f(n) = 1 + nf(n-1)$ for each positive integer $n$. Find the smallest prime divisor of $f(4 \times 30 + 2016)$.

**Solution.** For any prime number $p$ and nonnegative integer $k$ we can prove by induction on $r$ that

$$f(kp + r) \equiv f(r) \mod p$$

for any nonnegative integer $r$. The base case $r = 0$ follows from

$$f(kp) = 1 + kpf(kp-1) \equiv 1 \mod p$$

and $f(0) = 1$. The induction step "$r$ implies $r+1$" follows from

$$f(kp + r + 1) = 1 + (kp + r + 1)f(kp + r) \equiv 1 + (r+1)f(r) \mod p$$

and $f(r+1) = 1 + (r+1)f(r)$. Observe that $p$ could be also any positive integer. Since $N = 4 \times 30 + 2016 = 2136$ satisfies the following congruences

$$\begin{aligned} &N \equiv 0 \mod 2, && N \equiv 0 \mod 3, && N \equiv 1 \mod 5, \\ &N \equiv 1 \mod 7, && N \equiv 2 \mod 11, && N \equiv 4 \mod 13, \end{aligned}$$

and $f(0) = 1$, $f(1) = 2$, $f(2) = 5$, and $f(4) = 65 \equiv 0 \mod 13$, we conclude that $f(N)$ satisfies the following congruences

$$\begin{aligned} &f(N) \equiv 1 \mod 2, && f(N) \equiv 1 \mod 3, && f(N) \equiv 2 \mod 5, \\ &f(N) \equiv 2 \mod 7, && f(N) \equiv 5 \mod 11, && f(N) \equiv 0 \mod 13. \end{aligned}$$

Therefore the smallest prime divisor of $f(2136)$ is 13.