CCSU Regional Math Competition, 2018

SOLUTIONS I

Problem 1. Let S be the square in a rectangular coordinate plane with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Find a point P inside S such that the vertical line through P and the horizontal line through P split S into four regions whose areas form a (finite) geometric sequence with common ratio π.

Solution. If the coordinates of the point that we are trying to find are (x,y) with $0<x<y<\frac{1}{2}$ then the areas of the four rectangles will be xy, $x(1-y)$, $y(1-x)$, $(1-x)(1-y)$ (in increasing order). From $\dfrac{x(1-y)}{xy}=\dfrac{1-y}{y}=\pi$ we get $y=\dfrac{1}{\pi+1}$. From $\dfrac{y(1-x)}{x(1-y)}=\pi$ we get $\dfrac{1-x}{x\pi}=\pi$, hence $x=\dfrac{1}{\pi^2+1}$. Now we verify the final ratio: $\dfrac{(1-x)(1-y)}{y(1-x)}=\dfrac{1-y}{y}=\pi$. Notice that $\dfrac{1}{\pi^2+1}<\dfrac{1}{\pi+1}<\dfrac{1}{2}$.

Problem 2. Consider a 2×2 matrix A with real entries, whose determinant is $\det A=2$ and whose trace is $\mathrm{tr}A=2$. Show that

$$\det(A^2+xA+I)\geq \frac{1}{2}$$

for all real numbers x, where $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and for a matrix $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ the trace $\mathrm{tr}A=a+d$ and $\det A=ad-bc$.

Solution 1. Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, then $\det A=ad-bc=2$ and $\mathrm{tr}A=a+d=2$. Then

$$\begin{aligned}A^2+xA+I&=\begin{pmatrix}a^2+bc&ab+bd\\ac+dc&bc+d^2\end{pmatrix}+x\begin{pmatrix}a&b\\c&d\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\\&=\begin{pmatrix}a^2+bc+xa+1&ab+bd+xb\\ac+dc+cx&bc+d^2+xd+1\end{pmatrix}\\&=\begin{pmatrix}a(a+x)+bc+1&b(a+d+x)\\c(a+d+x)&d(d+x)+bc+1\end{pmatrix}\end{aligned}$$

Therefore $\det(A^2 + xA + I) =$

$$\begin{aligned}
&= [a(a+x) + bc + 1]\,[d(d+x) + bc + 1] - bc(a+d+x)^2 \\
&= ad(a+x)(d+x) + (bc+1)(a^2 + d^2 + ax + dx) + (bc+1)^2 - bc(a+d+x)^2 \\
&= ad(a+x)(d+x) + bc(a^2 + d^2 + ax + dx - a^2 - d^2 - x^2 - 2ad - 2ax - 2dx) \\
&+ a^2 + d^2 + ax + dx + (bc)^2 + 2bc + 1 \\
&= ad(a+x)(d+x) - bc(a+x)(d+x) - bc \cdot ad + a^2 + d^2 + ax + dx + (bc)^2 + 2bc + 1 \\
&= (ad - bc)(a+x)(d+x) - bc(ad - bc) + a^2 + d^2 + ax + dx + 2bc + 1 \\
&= 2(a+x)(d+x) - 2bc + a^2 + d^2 + ax + dx + 2bc + 1 \\
&= 2(a+x)(d+x) + a^2 + d^2 + ax + dx + 1 \\
&= 2ad + 2ax + 2dx + 2x^2 + a^2 + d^2 + ax + dx + 1 \\
&= (a+d)^2 + 3(a+d)x + 2x^2 + 1 \\
&= 1 + 2x^2 + 6x + 4 \\
&= 2x^2 + 6x + 5.
\end{aligned}$$

The function $f(x) = 2x^2 + 6x + 5$ has a minimum of $\frac{1}{2}$, at $x = -\frac{3}{2}$ (solve the equation $f'(x) = 0$).

Therefore

$$\det(A^2 + xA + I) = 2x^2 + 6x + 5 \geq \frac{1}{2}.$$

Solution 2 Consider the polynomial $p(t) = t^2 + xt + 1$, which has two roots t_1, t_2, having the property that $t_1 + t_2 = -x$ and $t_1 t_2 = 1$. Then the polynomial can be written as $p(t) = (t - t_1)(t - t_2)$. As a consequence, one has that the expression $A^2 + xA + I_2$ can be written as:

$$A^2 + xA + I = (A - t_1 I_2)(A - t_2 I_2)$$

Using the properties of the determinant, one obtains that:

$$\det(A^2 + xA + I) = \det(A - t_1 I)\det(A - t_2 I).$$

Given that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A - t_1 I = \begin{pmatrix} a - t_1 & b \\ c & d - t_1 \end{pmatrix}$ and $A - t_2 I = \begin{pmatrix} a - t_2 & b \\ c & d - t_2 \end{pmatrix}$. Their determinants are $(a - t_1)(d - t_1) - bc$ and $(a - t_1)(d - t_1) - bc$, respectively. Moreover, we are given that $\det A = ad - bc = 2$ and $\mathrm{tr} A = a + d = 2$.

Putting all together, one has the following:

$$
\begin{aligned}
\det(A^2+xA+I) &= \det(A-t_1I)\det(A-t_2I)\\
&= [(a-t_1)(d-t_1)-bc][(a-t_1)(d-t_1)-bc]\\
&= \left[ad-bc-(a+d)t_1+t_1^2\right]\left[ad-bc-(a+d)t_2+t_2^2\right]\\
&= \left(2-2t_1+t_1^2\right)\left(2-2t_2+t_2^2\right)\\
&= 4-4t_2+2t_2^2-4t_1+4t_1t_2-2t_1t_2^2+2t_1^2-2t_1^2t_2+t_1^2t_2^2\\
&= 4-4(t_1+t_2)-2t_1t_2(t_1+t_2)+2(t_1+t_2)^2+t_1^2t_2^2\\
&= 4-4(-x)-2(-x)+2x^2+1\\
&= 2x^2+6x+5,
\end{aligned}
$$

since $t_1+t_2=-x$ and $t_1t_2=1$.

The function $f(x)=2x^2+6x+5$ has a minimum of $\frac{1}{2}$, at $x=-\frac{3}{2}$ (solve the equation $f'(x)=0$).

Therefore

$$\det(A^2+xA+I_2)=2x^2+6x+5\geq\frac{1}{2}.$$

Problem 3. Consider the sequence of real numbers defined as

$$a_n=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}(\cos x)^{2n+1}\,dx.$$

Find the limit $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$.

Solution. We first obtain a recurrence relation between a_{n+1} and a_n. We need to use integration by parts:

$$
\begin{aligned}
a_{n+1} &= \frac{1}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+3}\ dx = \frac{1}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+2} (\sin x)'\ dx \\
&= \frac{1}{2^{2n+2}} (\cos x)^{2n+2} (\sin x) \Big|_0^{\frac{\pi}{2}} + \frac{2n+2}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} (\sin x)^2\ dx \\
&= \frac{2n+2}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} (\sin x)^2\ dx = \frac{n+1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} (1 - \cos^2 x)\ dx
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
a_{n+1} &= \frac{n+1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} (1 - \cos^2 x)\ dx \\
&= \frac{n+1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}\ dx - \frac{n+1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+3}\ dx \\
&= \frac{n+1}{2} a_n - 2(n+1) a_{n+1}
\end{aligned}
$$

This means that $(2n+3)a_{n+1} = \frac{n+1}{2} a_n$, from which it follows that $\frac{a_{n+1}}{a_n} = \frac{n+1}{4n+6}$, so

$$
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{4n+6} = \frac{1}{4}.
$$

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SOLUTIONS II

Problem 4. Find the limit

$$\lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}.$$

Solution. Solution. Since $\lim_{x\to 0}(\sin^{-1} x - \tan^{-1} x) = 0$ and $\lim_{x\to 0} x^3 = 0$ we can apply L'Hôpital's rule

$$\begin{aligned} L = \lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3} &= \lim_{x\to 0} \frac{1/\sqrt{1-x^2} - 1/(1+x^2)}{3x^2} \\ &= \lim_{x\to 0} \frac{1+x^2-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)} \\ &= \lim_{x\to 0} \frac{1+(1-\sqrt{1-x^2})/x^2}{3\sqrt{1-x^2}(1+x^2)}. \end{aligned}$$

The limit in the numerator is

$$\begin{aligned} \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2} &= \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \\ &= \lim_{x\to 0} \frac{x^2}{x^2(1+\sqrt{1-x^2})} \\ &= \lim_{x\to 0} \frac{1}{(1+\sqrt{1-x^2})} = \frac{1}{2}. \end{aligned}$$

The initial limit is thus $(1+\frac{1}{2})/3 = \frac{1}{2}$.

Problem 5. We define an operation $\circledcirc$ that interleaves two sequences as follows: For $A = a_1, a_2, \dots$ and $B = b_1, b_2, \dots$, let $A \circledcirc B = a_1, b_1, a_2, b_2, \dots$. Now, setting $C = 1, 0, 1, 0, \dots$, we define D to be the sequence such that $C \circledcirc D = D$. Find $\displaystyle\sum_{i=1}^{2018} D_i$.

Solution. Among the first 2018 terms of D there are 1009 in odd-index positions and 1009 in even-index positions. Those in odd positions are exactly

the first 1009 terms of sequence C, consisting of 505 ones and 504 zeroes, so their sum is 505. Those in even positions are identical to the first 1009 terms of D, so they split into 505 terms of C and 504 terms of D, etc. This iterative procedure eventually terminates, and yields the sum
$505 + 253 + 126 + 63 + 32 + 16 + 8 + 4 + 2 + 1 + 1 = 1011$.

Problem 6. Show that there exists a unique function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following two conditions:

a) $x^3 - (f(x))^3 + xf(x) = 0$ for all $x \in \mathbb{R}$;

b) $f(x) > 0$ for all $x > 0$.

Solution. Let $g(y) = x^3 - y^3 + xy$. Then $g'(y) = -3y^2 + x$ for all $y \in \mathbb{R}$. For $x = 0$ the equation $g(y) = -y^3 = 0$ has a unique solution $y = f(0) = 0$. For $x < 0$ the derivative $g'(y) = -3y^2 + x < 0$ for all $y \in \mathbb{R}$ so that g is decreasing from $\lim_{y\to-\infty} g(y) = +\infty$ to $\lim_{y\to\infty} g(y) = -\infty$ and thus, the equation $g(y) = 0$ has a unique solution $y = f(x)$ by the IVT. For $x > 0$ the derivative $g'(y) > 0$ for $0 < y < c$ and $g'(y) < 0$ for $y > c$ where $c = (x/3)^{1/2}$ is the only positive solution to the equation $g'(y) = -3y^2 + x = 0$. Hence, g is increasing from $g(0) = x^3$ to $g(c)$ and decreasing from $g(c)$ to $\lim_{y\to\infty} g(y) = -\infty$. Since $x^3 > 0$ there is only one solution $y = f(x) > 0$ to the equation $g(y) = 0$ by the IVT. This concludes the proof.