Problem 1. Let $S$ be the square in a rectangular coordinate plane with vertices $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$. Find a point $P$ inside $S$ such that the vertical line through $P$ and the horizontal line through $P$ split $S$ into four regions whose areas form a (finite) geometric sequence with common ratio $\pi$.

Solution. If the coordinates of the point that we are trying to find are $(x, y)$ with $0 < x < y < \frac{1}{2}$ then the areas of the four rectangles will be $xy$, $x(1-y)$, $y(1-x)$, $(1-x)(1-y)$ (in increasing order). From $\frac{x(1-y)}{xy} = \frac{1-y}{y} = \pi$ we get $y = \frac{1}{\pi + 1}$. From $\frac{y(1-x)}{x(1-y)} = \pi$ we get $\frac{1-x}{x\pi} = \pi$, hence $x = \frac{1}{\pi^2 + 1}$.

Now we verify the final ratio: $\frac{(1-x)(1-y)}{y(1-x)} = \frac{1-y}{y} = \pi$. Notice that $\frac{1}{\pi^2 + 1} < \frac{1}{\pi + 1} < \frac{1}{2}$.

Problem 2. Consider a $2 \times 2$ matrix $A$ with real entries, whose determinant is $\det A = 2$ and whose trace is $\text{tr}A = 2$. Show that

$$\det(A^2 + xA + I) \geq \frac{1}{2}$$

for all real numbers $x$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the trace $\text{tr}A = a + d$ and $\det A = ad - bc$.

Solution 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det A = ad - bc = 2$ and $\text{tr}A = a + d = 2$.

Then

$$A^2 + xA + I = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{pmatrix} + x \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + bc + xa + 1 & ab + bd + xb \\ ac + dc + cx & bc + d^2 + xd + 1 \end{pmatrix}$$

$$= \begin{pmatrix} a(a + x) + bc + 1 & b(a + d + x) \\ c(a + d + x) & d(d + x) + bc + 1 \end{pmatrix}$$
Therefore \( \det(A^2 + xA + I) = \)

\[
= [a(a + x) + bc + 1] [d(d + x) + bc + 1] - bc(a + d + x)^2 \\
= ad(a + x)(d + x) + (bc + 1)(a^2 + d^2 + ax + dx) + (bc + 1)^2 - bc(a + d + x)^2 \\
= ad(a + x)(d + x) + bc(a^2 + d^2 + ax + dx - a^2 - d^2 - x^2 - 2ad - 2ax - 2dx) \\
+ a^2 + d^2 + ax + dx + (bc)^2 + 2bc + 1 \\
= ad(a + x)(d + x) - bc(a + x)(d + x) - bc \cdot ad + a^2 + d^2 + ax + dx + (bc)^2 + 2bc + 1 \\
= (ad - bc)(a + x)(d + x) - bc(ad - bc) + a^2 + d^2 + ax + dx + 2bc + 1 \\
= 2(a + x)(d + x) - 2bc + a^2 + d^2 + ax + dx + 2bc + 1 \\
= 2(a + x)(d + x) + a^2 + d^2 + ax + dx + 1 \\
= 2ad + 2ax + 2dx + x^2 + a^2 + d^2 + ax + dx + 1 \\
= (a + d)^2 + 3(a + d)x + 2x^2 + 1 \\
= 1 + 2x^2 + 6x + 4 \\
= 2x^2 + 6x + 5.
\]

The function \( f(x) = 2x^2 + 6x + 5 \) has a minimum of \( \frac{1}{2} \), at \( x = -\frac{3}{2} \) (solve the equation \( f'(x) = 0 \)).

Therefore

\[
\det(A^2 + xA + I) = 2x^2 + 6x + 5 \geq \frac{1}{2}.
\]

**Solution 2** Consider the polynomial \( p(t) = t^2 + xt + 1 \), which has two roots \( t_1, t_2 \), having the property that \( t_1 + t_2 = -x \) and \( t_1t_2 = 1 \). Then the polynomial can be written as \( p(t) = (t - t_1)(t - t_2) \). As a consequence, one has that the expression \( A^2 + xA + I_2 \) can be written as:

\[
A^2 + xA + I = (A - t_1I_2)(A - t_2I_2)
\]

Using the properties of the determinant, one obtains that:

\[
\det(A^2 + xA + I) = \det(A - t_1I) \det(A - t_2I).
\]

Given that \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then \( A - t_1I = \begin{pmatrix} a - t_1 & b \\ c & d - t_1 \end{pmatrix} \) and \( A - t_2I = \begin{pmatrix} a - t_2 & b \\ c & d - t_2 \end{pmatrix} \). Their determinants are \((a - t_1)(d - t_1) - bc\) and \((a - t_1)(d - t_1) - bc\), respectively. Moreover, we are given that \( \det A = ad - bc = 2 \) and \( \text{tr}A = a + d = 2 \).
Putting all together, one has the following:

$$\det(A^2 + xA + I) = \det(A - t_1 I) \det(A - t_2 I)$$

$$= [(a - t_1)(d - t_1) - bc][a (a - t_1)(d - t_1) - bc]$$

$$= [ad - bc - (a + d)t_1 + t_1^2][ad - bc - (a + d)t_2 + t_2^2]$$

$$= (2 - 2t_1 + t_1^2)(2 - 2t_2 + t_2^2)$$

$$= 4 - 4t_2 + 2t_2^2 - 4t_1 + 4t_1 t_2 - 2t_1 t_2^2 + 2t_2^2 - 2t_1^2 t_2 + t_1^2 t_2^2$$

$$= 4 - 4(t_1 + t_2) - 2t_1 t_2(t_1 + t_2) + 2(t_1 + t_2)^2 + t_1^2 t_2^2$$

$$= 4 - 4(-x) - 2(-x) + 2x^2 + 1$$

$$= 2x^2 + 6x + 5,$$

since $t_1 + t_2 = -x$ and $t_1 t_2 = 1$.

The function $f(x) = 2x^2 + 6x + 5$ has a minimum of $\frac{1}{2}$, at $x = -\frac{3}{2}$ (solve the equation $f'(x) = 0$).

Therefore

$$\det(A^2 + xA + I_2) = 2x^2 + 6x + 5 \geq \frac{1}{2},$$

Problem 3. Consider the sequence of real numbers defined as

$$a_n = \frac{1}{2^n} \int_{0}^{\frac{\pi}{2}} (\cos x)^{2n+1} \, dx.$$

Find the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Solution. We first obtain a recurrence relation between $a_{n+1}$ and $a_n$. We
need to use integration by parts:

\[
a_{n+1} = \frac{1}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+3} \, dx = \frac{1}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+2}(\sin x)' \, dx
\]

\[
= \frac{1}{2^{2n+2}} (\cos x)^{2n+2}(\sin x) \bigg|_0^{\frac{\pi}{2}} + \frac{2n + 2}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}(\sin x)^2 \, dx
\]

\[
= \frac{2n + 2}{2^{2n+2}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}(\sin x)^2 \, dx = \frac{n + 1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}(1 - \cos^2 x) \, dx
\]

Therefore:

\[
a_{n+1} = \frac{n + 1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}(1 - \cos^2 x) \, dx
\]

\[
= \frac{n + 1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} \, dx - \frac{n + 1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (\cos x)^{2n+3} \, dx
\]

\[
= \frac{n + 1}{2} a_n - 2(n + 1)a_{n+1}
\]

This means that \((2n + 3)a_{n+1} = \frac{n+1}{2} a_n\), from which it follows that \(\frac{a_{n+1}}{a_n} = \frac{n+1}{4n+6}\), so

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n + 1}{4n + 6} = \frac{1}{4}.
\]
Problem 4. Find the limit
\[
\lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}.
\]

Solution. Since \(\lim_{x \to 0}(\sin^{-1} x - \tan^{-1} x) = 0\) and \(\lim_{x \to 0} x^3 = 0\) we can apply L'Hôpital’s rule

\[
L = \lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3} = \lim_{x \to 0} \frac{1/\sqrt{1-x^2} - 1/(1+x^2)}{3x^2} = \lim_{x \to 0} \frac{1+x^2 - \sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)} = \lim_{x \to 0} \frac{1+(1-\sqrt{1-x^2})/x^2}{3\sqrt{1-x^2}(1+x^2)}.
\]

The limit in the numerator is

\[
\lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x^2} = \lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x^2} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim_{x \to 0} \frac{x^2}{x^2(1+\sqrt{1-x^2})} = \lim_{x \to 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}.
\]

The initial limit is thus \((1 + \frac{1}{2})/3 = \frac{1}{2}\).

Problem 5. We define an operation \(\odot\) that interleaves two sequences as follows: For \(A = a_1, a_2, \ldots\) and \(B = b_1, b_2, \ldots\), let \(A \odot B = a_1, b_1, a_2, b_2, \ldots\). Now, setting \(C = 1, 0, 1, 0, \ldots\), we define \(D\) to be the sequence such that \(C \odot D = D\). Find \(\sum_{i=1}^{2018} D_i\).

Solution. Among the first 2018 terms of \(D\) there are 1009 in odd-index positions and 1009 in even-index positions. Those in odd positions are exactly
the first 1009 terms of sequence $C$, consisting of 505 ones and 504 zeroes, so their sum is 505. Those in even positions are identical to the first 1009 terms of $D$, so they split into 505 terms of $C$ and 504 terms of $D$, etc. This iterative procedure eventually terminates, and yields the sum $505 + 253 + 126 + 63 + 32 + 16 + 8 + 4 + 2 + 1 + 1 = 1011$.

**Problem 6.** Show that there exists a unique function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following two conditions:

a) $x^3 - (f(x))^3 + xf(x) = 0$ for all $x \in \mathbb{R}$;

b) $f(x) > 0$ for all $x > 0$.

**Solution.** Let $g(y) = x^3 - y^3 + xy$. Then $g'(y) = -3y^2 + x$ for all $y \in \mathbb{R}$. For $x = 0$ the equation $g(y) = -y^3 = 0$ has a unique solution $y = f(0) = 0$. For $x < 0$ the derivative $g'(y) = -3y^2 + x < 0$ for all $y \in \mathbb{R}$ so that $g$ is decreasing from $\lim_{y \to -\infty} g(y) = +\infty$ to $\lim_{y \to \infty} g(y) = -\infty$ and thus, the equation $g(y) = 0$ has a unique solution $y = f(x)$ by the IVT. For $x > 0$ the derivative $g'(y) > 0$ for $0 < y < c$ and $g'(y) < 0$ for $y > c$ where $c = (x/3)^{1/2}$ is the only positive solution to the equation $g'(y) = -3y^2 + x = 0$. Hence, $g$ is increasing from $g(0) = x^3$ to $g(c)$ and decreasing from $g(c)$ to $\lim_{y \to \infty} g(y) = -\infty$. Since $x^3 > 0$ there is only one solution $y = f(x) > 0$ to the equation $g(y) = 0$ by the IVT. This concludes the proof.