CCSU Regional Math Competition, 2018

Part I

Each problem is worth ten points. Please be sure to use separate pages to write your solution for every problem.

Problem 1. Let $S$ be the square in a rectangular coordinate plane with vertices $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$. Find a point $P$ inside $S$ such that the vertical line through $P$ and the horizontal line through $P$ split $S$ into four regions whose areas form a (finite) geometric sequence with common ratio $\pi$.

Problem 2. Consider a $2 \times 2$ matrix $A$ with real entries, whose determinant is $\det A = 2$ and whose trace is $\text{tr}A = 2$. Show that

$$\det(A^2 + xA + I) \geq \frac{1}{2}$$

for all real numbers $x$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the trace $\text{tr}A = a + d$ and $\det A = ad - bc$.

Problem 3. Consider the sequence of real numbers defined as

$$a_n = \frac{1}{2^{2n}} \int_{0}^{\frac{\pi}{2}} (\cos x)^{2n+1} \ dx.$$ 

Find the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.
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Part II

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Problem 4. Find the limit
\[ \lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}. \]

Problem 5. We define an operation \( \odot \) that interleaves two sequences as follows: For \( A = a_1, a_2, \ldots \) and \( B = b_1, b_2, \ldots \), let \( A \odot B = a_1, b_1, a_2, b_2, \ldots \). Now, setting \( C = 1, 0, 1, 0, \ldots \), we define \( D \) to be the sequence such that \( C \odot D = D \). Find \( \sum_{i=1}^{2018} D_i \).

Problem 6. Show that there exists a unique function \( f : \mathbb{R} \to \mathbb{R} \) satisfying the following two conditions:
\[ a) \quad x^3 - (f(x))^3 + xf(x) = 0 \text{ for all } x \in \mathbb{R}; \]
\[ b) \quad f(x) > 0 \text{ for all } x > 0. \]